On the cyclotomic Dedekind embedding and the cyclic Wedderburn embedding

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Abstract

Let $n \geq 1$ and let p be a prime. Expand $j \in [0, p^n-1]\backslash(p)$ p-adically as $j = \sum_{s\geq 0} a_s p^s$ with $a_s \in [0, p-1]$. The $\#([0,j]\backslash(p))$th $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linear elementary divisor of the cyclotomic Dedekind embedding

$$\mathbf{Z}_{(p)}[\zeta_{p^n}] \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}[\zeta_{p^n}] \hookrightarrow \prod_{i\in(\mathbf{Z}/p^n)^*} \mathbf{Z}_{(p)}[\zeta_{p^n}]$$

has valuation

$$-1 + \sum_{s\geq 0} \left(a_s(s+1) - a_{s+1}(s+2)\right) p^s$$

at $1 - \zeta_{p^n}$. There is a similar result for the related cyclic Wedderburn embedding.

0 Introduction

0.0 Contents

0.1 Objectives

In this note, we consider an extension of discrete valuation rings $S \subseteq T$, with $s \in S$ and $t \in T$ generating the respective maximal ideals. Let K be the field of fractions of S, let L be the field of fractions of T and assume L over K to be finite galois of Galois group G of order $m = |G|$. Moreover, assume $T = S[\vartheta]$ for some $\vartheta \in T$. We restrict the Dedekind isomorphism

$$\begin{array}{rcl} L \otimes_K L & \xrightarrow{\sim} & \prod\limits_{\sigma \in G} L \\ x \otimes y & \longmapsto & (xy^\sigma)_{\sigma \in G}, \end{array}$$

the injectivity of which ensues from Dedekind's Lemma, to the locally integral situation

$$T \otimes_S T \hookrightarrow \prod_{\sigma \in G} T.$$

This embedding is no longer an isomorphism in general; in fact, the square of its T-linear determinant is just the discriminant of T over S. Being a T-linear map, we can ask for its elementary divisors, or, more precisely, for an elementary divisor diagonalization.

Once such a diagonalization obtained, we may use the image of $T \otimes_S T$ inside $\prod_{\sigma \in G} T$, instead of $T \otimes_S T$ itself, to calculate Hochschild (co)homology of T. For instance, we may ask for these (co)homology groups with coefficients in T, equipped with a galois twisted T-T-bimodule structure (the untwisted case being due to M. Larsen and A. Lindenstrauss [LL 92, 1.6.2] and to the Buenos Aires Cyclic Homology group [BACH 91, prop. 1.3]).

The cyclotomic Dedekind embedding shall be studied in more detail, where $T = \mathbf{Z}_{(p)}[\zeta_{p^n}]$ and $S = \mathbf{Z}_{(p)}$, for $n \geq 1$, p a prime, and where, in general for $m \geq 1$, ζ_m denotes a primitive mth root of unity over $\mathbf{Q}$.

This cyclotomic case leads to a related question, which is to give an elementary divisor diagonalization for the cyclic Wedderburn embedding. Given $m \geq 1$, and letting c_m denote a generator of the cyclic group C_m of m elements, this embedding results from restricting the rational Wedderburn isomorphism

$$\begin{array}{rcl} \mathbf{Q}(\zeta_m)C_m & \xrightarrow{\sim} & \prod\limits_{j\in[0,m-1]} \mathbf{Q}(\zeta_m) \\ c_m & \longmapsto & (\zeta_m^j)_{j\in[0,m-1]}, \end{array}$$

to the integral situation

$$\mathbf{Z}[\zeta_m]C_m \hookrightarrow \prod_{j\in[0,m-1]} \mathbf{Z}[\zeta_m],$$

which is likewise no longer an isomorphism for $m > 1$, the square of its $\mathbf{Z}[\zeta_m]$-linear determinant being $\pm m^m$. Now the cyclotomic Dedekind embedding may be viewed, considered as a diagram, as a quotient of the corresponding cyclic Wedderburn embedding.

This consideration in turn gives rise to the question for a closed description of the image of the cyclic Wedderburn embedding in the absolute case, i.e. of

$$\begin{array}{rcl} \mathbf{Z}C_m & \hookrightarrow & \prod\limits_{d\,|\,m} \mathbf{Z}[\zeta_d] \\ c_m & \longmapsto & (\zeta_d)_{d\,|\,m}. \end{array}$$

0.2 Results

0.2.1 Hochschild (co)homology with twisted coefficients

We choose a total ordering on G, $(\sigma_0, \ldots, \sigma_{m-1})$, in such a way that $\sigma_0 = 1_T$, and such that

$$\sum_{i\in[0,j-1]} v_t(\vartheta^{\sigma_j} - \vartheta^{\sigma_i}) \leq \sum_{i\in[0,j-1]} v_t(\vartheta^{\sigma_k} - \vartheta^{\sigma_i})$$

for each $j \in [0, m-1]$ and each $k \in [j+1, m-1]$, where v_t denotes the t-adic valuation. Let T_i denote T, considered as a T-T-bimodule with untwisted left multiplication and σ_i-twisted right multiplication by T. Let $\varphi := \sum_{k\in[1,m-1]} v_t(\vartheta - \vartheta^{\sigma_k})$.

The Hochschild homology of T, over the ground ring S and with coefficients in T_i, $i \in [0, m-1]$, is given in dimension $j \geq 0$ by

$$H_j(T, T_0; S) \simeq \begin{cases} T & \text{for } j = 0 \\ T/t^{\varphi}T & \text{for } j \text{ odd} \\ 0 & \text{for } j \text{ even, } j \geq 2 \end{cases}$$

and by

$$H_j(T, T_i; S) \simeq \begin{cases} 0 & \text{for } j \text{ odd} \\ T/(\vartheta^{\sigma_i} - \vartheta)T & \text{for } j \text{ even} \end{cases}$$

for $i \in [1, m-1]$. The Hochschild cohomology is given by

$$H^j(T, T_0; S) \;\simeq\; \begin{cases} T & \text{for } j = 0 \\ 0 & \text{for } j \text{ odd} \\ T/t^{\varphi}T & \text{for } j \text{ even, } j \geq 2 \end{cases}$$

and by

$$H^j(T, T_i; S) \;\simeq\; \begin{cases} T/(\vartheta^{\sigma_i} - \vartheta)T & \text{for } j \text{ odd} \\ 0 & \text{for } j \text{ even} \end{cases}$$

for $i \in [1, m-1]$ (1.17).

0.2.2 Elementary divisors of the cyclotomic Dedekind embedding

Let $n \geq 1$ and let p be a prime. Expand $j \in [0, p^n - 1] \backslash (p)$ p-adically as $j = \sum_{s \geq 0} a_s p^s$ with $a_s \in [0, p-1]$. The $\#([0, j] \backslash (p))$th $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linear elementary divisor of the cyclotomic Dedekind embedding

$$\begin{array}{ccccc} \mathbf{Z}_{(p)}[\zeta_{p^n}] & \otimes_{\mathbf{Z}_{(p)}} & \mathbf{Z}_{(p)}[\zeta_{p^n}] & \overset{\delta_{p^n}}{\hookrightarrow} & \prod_{i \in (\mathbf{Z}/p^n)^*} \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ \zeta_{p^n}^k & \otimes & \zeta_{p^n}^l & \longmapsto & (\zeta_{p^n}^{k+il}), \end{array}$$

where $k, l \in [0, p^{n-1}(p-1) - 1]$, has valuation

$$-1 + \sum_{s \geq 0} \left(a_s(s+1) - a_{s+1}(s+2) \right) p^s$$

at $1 - \zeta_{p^n}$ (2.7).

0.2.3 The cyclic Wedderburn embedding

Expand $j \in [0, p^n - 1]$ p-adically as $j = \sum_{s \geq 0} a_s p^s$ with $a_s \in [0, p-1]$. The $(j+1)$st $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linear elementary divisor of the localized cyclic Wedderburn embedding

$$\begin{array}{ccc} \mathbf{Z}_{(p)}[\zeta_{p^n}]C_{p^n} & \overset{\omega_{p^n}}{\hookrightarrow} & \prod_{j \in \mathbf{Z}/p^n} \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ c_{p^n} & \longmapsto & (\zeta_{p^n}^j)_{j \in \mathbf{Z}/p^n} \end{array}$$

has valuation

$$\sum_{s \geq 0} (a_s - a_{s+1})(s+1) p^s$$

at $1 - \zeta_{p^n}$ (3.15). We can render this result a bit more precise and a bit more general, in that we can diagonalize the cyclic Wedderburn embedding

$$\begin{array}{ccc} \mathbf{Z}[\zeta_m]C_m & \overset{\omega_m}{\hookrightarrow} & \prod_{j \in \mathbf{Z}/m} \mathbf{Z}[\zeta_m] \\ c_m & \longmapsto & (\zeta_m^j)_{j \in \mathbf{Z}/m} \end{array}$$

for $m \geq 1$, regardless whether $\mathbf{Z}[\zeta_m]$ is a principal ideal domain (3.8). For $l \in [0, m-1]$, the $(l+1)$th diagonal entry becomes

$$\frac{m\zeta_m^{(l^2)}}{\prod_{j\in[1,l]}(1-\zeta_m^j)}.$$

0.2.4 The absolute cyclic Wedderburn embedding

It suffices to consider the primary parts of m separately, i.e. we may assume $m = p^n$ (cf. 5.18). In this case, the absolute cyclic Wedderburn embedding is given by

$$\begin{array}{rcl} \mathbf{Z}C_{p^n} & \overset{\omega_{\mathbf{Z},p^n}}{\hookrightarrow} & \prod_{i\in[0,n]} \mathbf{Z}[\zeta_{p^i}] \\ c_{p^n} & \longmapsto & (\zeta_{p^i})_{i\in[0,n]}. \end{array}$$

We derive from the pullback of Kervaire and Murthy [KM 77] the following triangular system of ties, i.e. of congruences of tuple entries, that describe its image.

Let φ denote Euler's function. For $m \geq 1$ and $j \in \mathbf{Z}$, we let $[j]_m \in [0, m-1]$ be such that $[j]_m \equiv_m j$. Let ∂ denote Kronecker's delta. The image of the absolute Wedderburn embedding is given by

$$\begin{array}{rl} (\mathbf{Z}C_{p^n})\omega_{\mathbf{Z},p^n} = & \Bigg\{ \left(\sum\limits_{j\in[0,\varphi(p^i)-1]} x_{i,j}\zeta_{p^i}^j \right)_{i\in[0,n]},\ x_{i,j} \in \mathbf{Z} \ \Bigg| \\ & \text{for } l \in [1,n] \text{ and } j \in [0, \varphi(p^{n-l})-1] \text{ we have } x_{n-l,j} \equiv_{p^l} \sum\limits_{i\in[0,l-1]} p^{l-1-i} \cdot \\ & \cdot \sum\limits_{k\in[1,p-1]} \left(x_{n-i,j-p^{n-l}+kp^{n-1-i}} - (1-\partial_{l,n}) x_{n-i,[j]_{p^{n-l-1}}-p^{n-l-1}+kp^{n-1-i}} \right) \Bigg\} \\ \subseteq & \prod\limits_{i\in[0,n]} \mathbf{Z}[\zeta_{p^i}]. \end{array}$$

The elementary divisors of $\omega_{\mathbf{Z},p^n}$ over $\mathbf{Z}$ are given by p^i with multiplicity $\varphi(p^{n-i})$ for $i \in [0, n]$ (5.14).

Kleinert gives a system of ties that describes the image of the absolute Wedderburn embedding $\mathbf{Z}C_m \xrightarrow{\omega_{\mathbf{Z},m}} \prod_{d|m} \mathbf{Z}[\zeta_d]$ in terms of certain prime ideals of the rings $\mathbf{Z}[\zeta_d]$, $d|m$, in case m is a positive squarefree integer [Kl 81, p. 550]. In loc. cit., this description is used as a tool to study units in dihedral group rings.

0.3 Acknowledgements

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Notation 0.1

(o) Composition of maps is written on the right, $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$.

(i) If φ is a map of A-modules, A a commutative ring, and $\mathfrak{p}$ a prime ideal of A, we sometimes denote the localization $\varphi_{\mathfrak{p}}$ merely by φ. If $m \geq 1$, we denote the ring of $m \times m$-matrices with entries in A by $(A)_m$.

(ii) If x, y are elements of some set, we let $\partial_{x,y} := 1$ if $x = y$ and $\partial_{x,y} := 0$ if $x \neq y$.

(iii) For integers a, b, we denote $[a, b] := \{x \in \mathbf{Z} \mid a \leq x \leq b\}$ and $[a, b[:= \{x \in \mathbf{Z} \mid a \leq x < b\}$.

(iv) For $a, b \geq 0$, we let the binomial coefficient $\binom{a}{b}$ be equal to zero if $b > a$.

(v) Let $\otimes = \otimes_{\mathbf{Z}}$ (i.e. $\otimes = \otimes_{\mathbf{Q}}$ over $\mathbf{Q}$).

Let $m \geq 1$.

(vi) For a prime p, we let $m[p] := p^{v_p(m)}$ denote the p-part of m.

(vii) Let φ denote Euler's function, $\varphi(m) := m\prod_{p|m}(1-p^{-1})$. Let $\Phi_m(X)$ denote the m-th cyclotomic polynomial, i.e. the irreducible factor of $X^m - 1 \in \mathbf{Q}[X]$ that does not divide $X^{m'} - 1$ for any $m'|m$, $m' \neq m$. Hence $\deg \Phi_m(X) = \varphi(m)$.

(viii) Let ζ_m denote a primitive mth root of unity over $\mathbf{Q}$, with minimal polynomial $\mu_{\zeta_m,\mathbf{Q}}(T) = \Phi_m(T)$. Let c_m denote a generator of the cyclic group C_m of m elements.

1 Vandermonde

We diagonalize the polynomial Vandermonde matrix rationally. Specializing under the assumption that the diagonalizing matrices become integral, we obtain an elementary divisor form, i.e. a diagonal matrix in which each subsequent diagonal entry is divisible by its predecessor.

1.1 Diagonalization of Vandermonde matrices

A hint of T. Zink how to simplify our previous efforts led to the approach in this and in the next subsection.

Notation 1.1 Let $m \geq 1$. Let $x = (x_0, \ldots, x_{m-1})$ be a tuple of indeterminates, let $a, b \in [0, m-1]$ and let $d \in \mathbf{Z}$. We consider in $\mathbf{Z}[x_0, \ldots, x_{m-1}]$ the symmetric polynomials

$$P_{d,[a,b]} := \begin{cases} \sum_{s_i \geq 0,\, \sum_{i\in[a,b]} s_i = d} x_a^{s_a} x_{a+1}^{s_{a+1}} \cdots x_b^{s_b} & \text{if } d \geq 0 \\ 0 & \text{if } d < 0 \end{cases}$$

$$E_{d,[a,b[} := \begin{cases} \sum_{s_i \in\{0,1\},\, \sum_{i\in[a,b[} s_i = d} x_a^{s_a} x_{a+1}^{s_{a+1}} \cdots x_{b-1}^{s_{b-1}} & \text{if } d \geq 0 \\ 0 & \text{if } d < 0. \end{cases}$$

In particular, we have

$$\begin{aligned} P_{d,\emptyset} &= \partial_{d,0} \\ E_{d,\emptyset} &= \partial_{d,0}. \end{aligned}$$

Moreover, for $i \in [0, m-1]$ we denote

$$y_i := \prod_{k\in[0,i[} (x_i - x_k) = \sum_{j\in[0,i]} (-1)^{i-j} E_{i-j,[0,i[} x_i^j,$$

considered as element of $\mathbf{Z}[x_0, \ldots, x_{m-1}]$. For $i, j \in [0, m-1]$, we denote the Lagrange interpolation function by

$$L_{i,j} := \begin{cases} \dfrac{\prod_{k\in[0,j[\setminus\{i\}}(x_j - x_k)}{\prod_{k\in[0,j[\setminus\{i\}}(x_i - x_k)} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases},$$

and as a variant

$$M_{i,j} := \frac{\prod_{k\in[0,i[}(x_j - x_k)}{\prod_{k\in[0,i[}(x_i - x_k)},$$

considered as elements of $\mathbf{Q}(x_0, \ldots, x_{m-1})$. These polynomials furnish the matrices

$$\begin{aligned} I &:= (\partial_{i,j})_{i\times j\in[0,m-1]\times[0,m-1]} \\ V_x &:= (x_j^i)_{i\times j\in[0,m-1]\times[0,m-1]} \\ L_x &:= (L_{i,j})_{i\times j\in[0,m-1]\times[0,m-1]} \\ M_x &:= (M_{i,j})_{i\times j\in[0,m-1]\times[0,m-1]} \\ P_x &:= (P_{i-j,[0,j]})_{i\times j\in[0,m-1]\times[0,m-1]} \\ E_x &:= ((-1)^{i-j} E_{i-j,[0,i[})_{i\times j\in[0,m-1]\times[0,m-1]} \\ Y_x &:= (\partial_{i,j} y_i)_{i\times j\in[0,m-1]\times[0,m-1]} \end{aligned}$$

in $(\mathbf{Q}(x_0, \ldots, x_{m-1}))_m$.

Lemma 1.2 *For $0 \leq a \leq b \leq m$ and $d \in \mathbf{Z}$, we have*

$$\begin{aligned} P_{d,[a,b]} &= P_{d,[a+1,b]} + x_a P_{d-1,[a,b]} &&= P_{d,[a,b-1]} + x_b P_{d-1,[a,b]} \\ E_{d,[a,b[} &= E_{d,[a+1,b[} + x_a E_{d-1,[a+1,b[} &&= E_{d,[a,b-1[} + x_b E_{d-1,[a,b-1[}, \end{aligned}$$

where for the second equation, we stipulate in addition that $a < b$.

Lemma 1.3 *For $i, k \in \mathbf{Z}$ and $0 \leq a \leq b < c \leq m$, we obtain*

$$\sum_{j \in \mathbf{Z}} (-1)^j E_{i-j,[a,c[} P_{j-k,[a,b]} = (-1)^k E_{i-k,[b+1,c[}.$$

In fact,

$$\begin{array}{rcl} \sum_{j\in\mathbf{Z}} (-1)^j E_{k-j,[a,c[} P_{j-i,[a,b]} & \stackrel{(1.2)}{=} & \sum_{j\in\mathbf{Z}} (-1)^j E_{k-j,[a+1,c[} P_{j-i,[a,b]} \\ & + & \sum_{j\in\mathbf{Z}} (-1)^j x_a E_{k-j-1,[a+1,c[} P_{j-i,[a,b]} \\ & = & \sum_{j\in\mathbf{Z}} (-1)^j E_{k-j,[a+1,c[} P_{j-i,[a,b]} \\ & + & \sum_{j\in\mathbf{Z}} (-1)^{j-1} E_{k-j,[a+1,c[} x_a P_{j-1-i,[a,b]} \\ & \stackrel{(1.2)}{=} & \sum_{j\in\mathbf{Z}} (-1)^j E_{k-j,[a+1,c[} P_{j-i,[a,b]} \\ & + & \sum_{j\in\mathbf{Z}} (-1)^{j-1} E_{k-j,[a+1,c[} P_{j-i,[a,b]} \\ & - & \sum_{j\in\mathbf{Z}} (-1)^{j-1} E_{k-j,[a+1,c[} P_{j-i,[a+1,b]} \\ & = & \sum_{j\in\mathbf{Z}} (-1)^j E_{k-j,[a+1,c[} P_{j-i,[a+1,b]}. \end{array}$$

Lemma 1.4 *We have $E_x P_x = I$.*

Given $i, k \in [0, m-1]$, we may assume $i > k$ and obtain

$$\begin{array}{rcl} (E_x P_x)_{i,k} & = & \sum_{j\in[0,m-1]} (-1)^{i-j} E_{i-j,[0,i[} P_{j-k,[0,k]} \\ & \stackrel{(1.3)}{=} & (-1)^{i-k} E_{i-k,[k+1,i[} \\ & = & 0. \end{array}$$

Lemma 1.5 *We have $M_x(I - L_x) = I$.*

Given $i, k \in [0, m-1]$, we may assume $i < k$ and need to show that

$$M_{i,k} - \sum_{j \in [i,k[} M_{i,j} L_{j,k} = 0.$$

This expression being an element of $\mathbf{Q}(x_0, \ldots, x_{k-1}, x_{k+1}, x_{m-1})[x_k]$ of degree in x_k less than k, we are reduced to plug in $x_k = x_l$ for $l \in [0, k-1]$.

Lemma 1.6 *We have $E_x V_x = Y_x M_x$.*

Given $i, k \in [0, m-1]$, we have

$$\sum_{j \in [0,m-1]} (-1)^{i-j} E_{i-j,[0,i[} x_k^j = \prod_{j \in [0,i[} (x_k - x_j) = y_i M_{i,k}.$$

Proposition 1.7 *We have $E_x V_x (I - L_x) = Y_x$.*

This follows from (1.5, 1.6).

Remark 1.8 *We have $V_x = P_x Y_x M_x$.*

This follows from (1.4, 1.6).

Example 1.9 Letting $m = 4$, we obtain

$$
\begin{aligned}
& E_x V_x (I - L_x) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0 x_1 & -(x_0+x_1) & 1 & 0 \\ -x_0 x_1 x_2 & x_0 x_1 + x_0 x_2 + x_1 x_2 & -(x_0+x_1+x_2) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -\frac{x_2 - x_1}{x_0 - x_1} & -\frac{(x_3-x_1)(x_3-x_2)}{(x_0-x_1)(x_0-x_2)} \\ 0 & 1 & -\frac{x_2-x_0}{x_1-x_0} & -\frac{(x_3-x_0)(x_3-x_2)}{(x_1-x_0)(x_1-x_2)} \\ 0 & 0 & 1 & -\frac{(x_3-x_0)(x_3-x_1)}{(x_2-x_0)(x_2-x_1)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (x_1 - x_0) & 0 & 0 \\ 0 & 0 & (x_2-x_0)(x_2-x_1) & 0 \\ 0 & 0 & 0 & (x_3-x_0)(x_3-x_1)(x_3-x_2) \end{bmatrix} = Y_x.
\end{aligned}
$$

1.2 Elementary divisors of Vandermonde matrices over discrete valuation rings

Setup 1.10 Let T be a discrete valuation ring with fraction field $L = \operatorname{frac} T$. Let $t \in T$ be a generator of the maximal ideal, and let v_t be the according valuation.

Definition 1.11 Let $m \geq 1$. Given a tuple $\xi = (\xi_0, \ldots, \xi_{m-1})$ of pairwise distinct elements of T, we say that ξ is *minimally ordered,* if

$$\sum_{i \in [0,j-1]} v_t(\xi_j - \xi_i) \leq \sum_{i \in [0,j-1]} v_t(\xi_k - \xi_i)$$

for each $j \in [0, m-1]$ and each $k \in [j+1, m-1]$. Note that any tuple of pairwise distinct elements of T can be reordered (non-uniquely, in general) to a minimally ordered tuple.

Lemma 1.12 *If ξ is minimally ordered, then $L_\xi \in (T)_m$ (cf. 1.1). In particular, the $(i+1)$st T-linear elementary divisor of V_ξ, where $i \in [0, m-1]$, has valuation*

$$\sum_{j \in [0,i-1]} v_t(\xi_i - \xi_j).$$

By (1.5), L_ξ is contained in $(T)_m$ if and only if M_ξ is contained in $(T)_m$. The assertion on the elementary divisors now ensues from (1.7).

Setup 1.13 Let $S \subseteq T$ be a finite extension of discrete valuation rings, with fraction fields $K := \operatorname{frac} S$ and $L := \operatorname{frac} T$. Let $s \in S$ and $t \in T$ be generators of the respective maximal ideals, and v_s and v_t the respective valuations. Assume L over K to be galois of degree $m = [L : K]$ with Galois group G. Assume that

$$T = S[\vartheta]$$

for some $\vartheta \in T$. Choose an ordering

$$
\begin{aligned}
[0, m-1] &\xrightarrow{\sim} G \\
i &\longmapsto \sigma_i
\end{aligned}
$$

such that the tuple $\tau := (\vartheta^{\sigma_1}, \ldots, \vartheta^{\sigma_{m-1}})$ is minimally ordered, starting with $\sigma_0 = 1_T$.

Proposition 1.14 *For $i, j \in [0, m-1]$, we let $L_{i,j}(\tau)$ denote the specialization of $L_{i,j}$ along $x \longmapsto \tau$ (cf. 1.1). The following assertions hold.*

(i) The $(i+1)$st T-linear elementary divisor, $i \in [0, m-1]$, of the Dedekind embedding

$$\begin{array}{ccccc} T & \otimes_S & T & \overset{\delta_{T/S}}{\hookrightarrow} & \prod_{\sigma\in G} T \\ x & \otimes & y & \longmapsto & (xy^\sigma)_{\sigma\in G} \end{array}$$

has valuation

$$\varphi_i := \sum_{j\in[0,i-1]} v_t(\vartheta^{\sigma_i} - \vartheta^{\sigma_j}).$$

In particular, $(\varphi_i)_{i\in[0,m-1]}$ does neither depend on the choice of ϑ nor on the minimal ordering chosen on τ.

(ii) The image of $T \otimes_S T$ under $\delta_{T/S}$, which is an isomorphic copy of $T \otimes_S T$, allows a description via ties, i.e. via congruences of tuple entries, as

$$\boxed{\begin{array}{l} (T \otimes_S T)\delta_{T/S} \\ \quad = \left\{ (\eta_j)_{j\in[0,m-1]} \;\middle|\; \eta_i - \sum_{j\in[0,i-1]} \eta_j L_{j,i}(\tau) \in Tt^{\varphi_i} \;\text{ for } i \in [0, m-1] \right\} \\ \quad \subseteq \prod_{j\in[0,m-1]} T. \end{array}}$$

(iii) A T-linear basis of $(T \otimes_S T)\delta_{T/S}$ of triangular shape is given by the following tuple of elements of $\prod_{j\in[0,m-1]} T$.

$$\left(\left(\prod_{k\in[0,i-1]} (\vartheta^{\sigma_j} - \vartheta^{\sigma_k}) \right)_{j\in[0,m-1]} \right)_{i\in[0,m-1]} .$$

Ad (i). This follows from (1.12).

Ad (ii). An element η of $\prod_{i\in[0,m-1]} T$, considered as a row vector, is contained in the image of $\delta_{T/S}$ if and only if its rational inverse image, written as a row vector in the L-linear basis $(1 \otimes \vartheta^0, \ldots, 1 \otimes \vartheta^{m-1})$ of $L \otimes_K L$, is in $T \otimes_S T$, i.e. if and only if ηV_τ^{-1} has entries in T. Which, in turn, is equivalent to $\eta(I - L_\tau)Y_\tau^{-1}$ having entries in T (1.7).

Ad (iii). We use $E_\tau V_\tau = Y_\tau M_\tau$ (1.6).

1.3 A projective resolution of T over $T \otimes_S T$

In the introduction to [LL 92], several sources for a projective resolution of T over $T \otimes_S T$ are indicated. We give still another alternative way to view such a projective resolution, using our isomorphic copy $(T \otimes_S T)\delta_{T/S}$. We include the case of Hochschild (co)homology with galois twisted coefficients, for there a shift in the 2-periodic vanishing of these groups occurs when compared to the untwisted case.

Notation 1.15 We denote $\Lambda := (T \otimes_S T)\delta_{T/S}$, i.e.

$$T \otimes_S T \xrightarrow[\sim]{\delta_{T/S}} \Lambda \hookrightarrow \prod_{\sigma \in G} T.$$

For $i \in [0, m-1]$, we dispose of the projection map

$$\begin{array}{rcl} \Lambda & \xrightarrow{\pi_i} & T \\ (\eta_j)_{j \in [0,m-1]} & \longmapsto & \eta_i, \end{array}$$

which is a ring morphism, and by means of which T becomes a module over Λ, denoted by T_i.

If we identify along $\delta_{T/S}$, the module operation of $x \otimes y \in T \otimes_S T$ on $z \in T_i$ is given by $z \cdot (x \otimes y) := xzy^{\sigma_i}$. I.e. T_i may be viewed as T equipped with a structure as a galois twisted T-T-bimodule. In particular, T_0 may be viewed as T equipped with the structure as an untwisted T-T-bimodule, that is $z \cdot (x \otimes y) := xzy$. Note that π_0 is just the multiplication map.

In Λ, we have the elements

$$\begin{array}{rcl} a & := & (\prod_{k \in [1,m-1]}(\vartheta - \vartheta^{\sigma_k}), 0, \dots, 0) \\ b & := & (0, \vartheta^{\sigma_1} - \vartheta, \dots, \vartheta^{\sigma_{m-1}} - \vartheta). \end{array}$$

The element a is in Λ by (1.14 ii), and b is in Λ by (1.14 iii). The multiplication map by a resp. by b shall be denoted by $\Lambda \xrightarrow{\alpha} \Lambda$ resp. by $\Lambda \xrightarrow{\beta} \Lambda$.

Lemma 1.16 *We have a 2-periodic projective resolution*

$$\cdots \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} \Lambda \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} \Lambda \xrightarrow{\pi_0} T_0$$

of T_0 over Λ.

From $ab = 0$ we take $\alpha\beta = 0$ and $\beta\alpha = 0$. Moreover, $\beta\pi_0 = 0$. By (1.14 ii), we obtain that the image of α equals the kernel of β. To prove that the image of β equals the kernel of α (resp. of π_0), means, after identification along $\delta_{T/S}$, to show that this kernel is $T \otimes_S T$-linearly generated by $b = 1 \otimes \vartheta - \vartheta \otimes 1$. In fact, suppose given $\sum_{i \in [0,m-1]} u_i \otimes \vartheta^i$ such that $\sum_{i \in [0,m-1]} u_i \vartheta^i = 0$, we may write $\sum_{i \in [0,m-1]} u_i \otimes \vartheta^i = \sum_{i \in [0,m-1]} u_i(1 \otimes \vartheta^i - \vartheta^i \otimes 1)$, and $1 \otimes \vartheta^i - \vartheta^i \otimes 1$ is a multiple of $1 \otimes \vartheta - \vartheta \otimes 1$.

Proposition 1.17 (cf. [BACH 91, prop. 1.3], [LL 92, 1.6.2])
Let $\varphi := \sum_{k \in [1,m-1]} v_t(\vartheta - \vartheta^{\sigma_k})$. The Hochschild homology of T, over the ground ring S and with coefficients in T_i, $i \in [0, m-1]$, is given in dimension $j \geq 0$ by

$$H_j(T, T_0; S) \simeq \begin{cases} T & \text{for } j = 0 \\ T/t^{\varphi}T & \text{for } j \text{ odd} \\ 0 & \text{for } j \text{ even, } j \geq 2 \end{cases}$$

and by

$$H_j(T, T_i; S) \simeq \begin{cases} 0 & \textit{for } j \textit{ odd} \\ T/(\vartheta^{\sigma_i} - \vartheta)T & \textit{for } j \textit{ even} \end{cases}$$

for $i \in [1, m-1]$. *The Hochschild cohomology is given by*

$$H^j(T, T_0; S) \simeq \begin{cases} T & \textit{for } j = 0 \\ 0 & \textit{for } j \textit{ odd} \\ T/t^{\varphi}T & \textit{for } j \textit{ even, } j \geq 2 \end{cases}$$

and by

$$H^j(T, T_i; S) \simeq \begin{cases} T/(\vartheta^{\sigma_i} - \vartheta)T & \textit{for } j \textit{ odd} \\ 0 & \textit{for } j \textit{ even} \end{cases}$$

for $i \in [1, m-1]$.

Tensoring the resolution in (1.16) with T_i over Λ yields the homology. Application of $\mathrm{Hom}_\Lambda(-, T_i)$ yields the cohomology.

1.4 The local ring $T \otimes_S T$

Assume $\vartheta = t$.

Remark 1.18 The radical of Λ is given by $\mathfrak{r}\Lambda = \Lambda \cap \prod_{i\in[0,m-1]} tT$. In fact, the $(\varphi+1)$st power of this intersection is contained in $t\Lambda$ (1.14 ii). And conversely, by (1.14 iii), this intersection has the T-linear basis

$$\Big\{(t, \dots, t)\Big\} \cup \Big\{ \Big(\textstyle\prod_{k\in[0,i-1]} (t^{\sigma_j} - t^{\sigma_k}) \Big)_{j\in[0,m-1]} \;\Big|\; i \in [1, m-1] \Big\},$$

whence $\Lambda/(\Lambda \cap \prod_{i\in[0,m-1]} tT)$ is isomorphic to T_0/tT_0. In particular, $T \otimes_S T$ is a local ring. Identifying along $\delta_{T/S}$, a T-linear basis of the radical $\mathfrak{r}\Lambda$ is given by

$$(t \otimes 1, 1 \otimes t, 1 \otimes t^2, \dots, 1 \otimes t^{m-1}).$$

Given $j \in \mathbf{Z}$, we write $\underline{j} := \max(j, 0)$. Suppose given $i \geq 0$. A basis of $\mathfrak{r}^i\Lambda$ is given by

$$(t^{\underline{i-0}} \otimes 1, t^{\underline{i-1}} \otimes t, t^{\underline{i-2}} \otimes t^2, \dots, t^{\underline{i-(m-1)}} \otimes t^{m-1}).$$

In particular, we have

$$\dim_{T/tT} \mathfrak{r}^i\Lambda/\mathfrak{r}^{i+1}\Lambda = \min(i+1, m).$$

Cf. [Kü 99, E.2.3].

2 The cyclotomic Dedekind embedding

We shall apply (1.14) to the case of the extension $\mathbf{Z}_{(p)} \subseteq \mathbf{Z}_{(p)}[\zeta_{p^n}]$.

Setup 2.1 Let p be a prime, let $n \geq 1$. In the notation of (1.13), we place ourselves in the situation $S = \mathbf{Z}_{(p)}$, $s = p$, $T = \mathbf{Z}_{(p)}[\zeta_{p^n}]$ and $t = \vartheta = 1 - \zeta_{p^n}$.

We consider the Dedekind embedding

$$\begin{array}{ccccc} \mathbf{Z}_{(p)}[\zeta_{p^n}] & \otimes & \mathbf{Z}_{(p)}[\zeta_{p^n}] & \xhookrightarrow{\delta_{p^n}} & \prod_{j\in(\mathbf{Z}/p^n)^*} \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ \zeta_{p^n}^k & \otimes & \zeta_{p^n}^l & \longmapsto & (\zeta_{p^n}^k \zeta_{p^n}^{jl})_{j\in(\mathbf{Z}/p^n)^*} \end{array}$$

where $k, l \in [0, p^n - 1]$. With respect to the $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linear basis $(1 \otimes t^i)_{i\in[0,(p-1)p^{n-1}-1]}$ of $\mathbf{Z}_{(p)}[\zeta_{p^n}] \otimes \mathbf{Z}_{(p)}[\zeta_{p^n}]$ and to the tuple basis on the right hand side, this embedding is given by the Vandermonde matrix V_τ for $\tau := (1 - \zeta_{p^n}^j)_{j\in[0,p^n-1]\setminus(p)}$.

Lemma 2.2 *For $i, j \in [0, p^n - 1]$ with $i \neq j$, we have $v_t(\zeta_{p^n}^i - \zeta_{p^n}^j) = (i - j)[p]$.*

We may assume $j = 0$ and $i = i[p]$. Using $v_t(p) = (p-1)p^{n-1}$ (resp. using a direct calculation if $p = 2$, $i = 2^{n-1}$), the congruence $\zeta_{p^n}^i - 1 \equiv_p (\zeta_{p^n} - 1)^i$ yields $v_t(\zeta_{p^n}^i - 1) = v_t((\zeta_{p^n} - 1)^i) = i$.

Lemma 2.3 *Suppose given $j \geq 0$, and write it p-adically as $j = \sum_{l\geq 0} a_l p^l$, where $a_l \in [0, p-1]$. Then*

$$\sum_{i\in[1,j]} i[p] \;=\; \sum_{k\geq 0} (a_k - a_{k+1})(k+1)p^k.$$

Denote left and right hand side of the claimed equation by $l(j)$ and $r(j)$, respectively. We have $l(0) = 0 = r(0)$. Moreover, for $j \geq 1$ we have $l(j) - l(j-1) = j[p]$. If $v_p(j) = 0$, then $r(j) - r(j-1) = 1$, so that we may suppose $v_p(j) \geq 1$. Writing $j - 1 = \sum_{k\geq 0} a'_k p^k$, $a'_k \in [0, p-1]$, we note that $a'_k = p - 1$ and $a_k = 0$ for $k \leq v_p(j) - 1$, $a'_k = a_k - 1$ for $k = v_p(j)$ and $a'_k = a_k$ for $k \geq v_p(j) + 1$. Therefore,

$$\begin{aligned} r(j) - r(j-1) &= \sum_{k\geq 0} (a_k - a'_k)(k+1)p^k - \sum_{k\geq 1} (a_k - a'_k) k p^{k-1} \\ &= (v_p(j) + 1) j[p] \;-\; \sum_{k\in[0,v_p(j)-1]} (p-1)(k+1)p^k \\ &\quad -\; v_p(j) j[p]/p \;+\; \sum_{k\in[0,v_p(j)-2]} (p-1)(k+1)p^k \\ &= j[p]. \end{aligned}$$

Lemma 2.4 *Suppose given $k \geq j \geq 0$, with $j, k \notin (p)$. Write $j = \sum_{l\geq 0} a_l p^l$, $k = \sum_{l\geq 0} a'_l p^l$, $k - j = \sum_{l\geq 0} a''_l p^l$, where $a_l, a'_l, a''_l \in [0, p-1]$. Then*

$$\sum_{i\in[0,j-1]\setminus(p)} (k-i)[p] = -1 + \sum_{l\geq 0} \Big(\big((a'_l - a''_l) - (a'_{l+1} - a''_{l+1}) \big)(l+1) - a_{l+1} \Big) p^l.$$

In particular,

$$\sum_{i\in[0,j-1]\backslash(p)} (j-i)[p] = -1 + \sum_{l\geq 0} \left(a_l(l+1) - a_{l+1}(l+2)\right) p^l.$$

This follows by (2.3) and the remark that $\sum\limits_{i\in[0,j-1]\cap(p)} (k-i)[p] = \#([0,j-1]\cap(p)) = 1 + \sum\limits_{l\geq 0} a_{l+1}p^l$.

Lemma 2.5 *Keep the notation of (2.4), but allow $k \geq j \geq 0$ to be arbitrary. We have*

$$\sum_{l\geq 0} \Big((a'_l - a''_l - a_l) - (a'_{l+1} - a''_{l+1} - a_{l+1})\Big)\,(l+1)p^l \geq 0.$$

Let $U := \{l \geq 0 \mid a_l + a''_l \geq p\} \subseteq \mathbf{Z}_{\geq 0}$. We obtain

$$\begin{aligned} & \sum_{l\geq 0} \Big((a'_l - a''_l - a_l) - (a'_{l+1} - a''_{l+1} - a_{l+1})\Big)\,(l+1)p^l \\ = & \sum_{l\in U} (-p)(l+1)p^l + \sum_{l\in U+1} (l+1)p^l - \sum_{l\in U-1} (-p)(l+1)p^l - \sum_{l\in U} (l+1)p^l \\ = & \sum_{l\in U} \Big(-(l+1)p^{l+1} + (l+2)p^{l+1} + lp^l - (l+1)p^l\Big) \\ = & \sum_{l\in U} (p-1)p^l. \end{aligned}$$

Lemma 2.6 *The tuple $\tau = (1-\zeta_{p^n}^j)_{j\in[0,p^n-1]\backslash(p)}$ is minimally ordered.*

Using (2.2), we need to see that

$$\sum_{i\in[0,j-1]\backslash(p)} (j-i)[p] \leq \sum_{i\in[0,j-1]\backslash(p)} (k-i)[p]$$

for $j \in [0,p^n-1]\backslash(p)$ and $k \in [j+1,p^n-1]\backslash(p)$. In the notation and using the assertion of (2.4) this amounts to the inequality

$$\sum_{l\geq 0} \Big((a'_l - a''_l - a_l) - (a'_{l+1} - a''_{l+1} - a_{l+1})\Big)\,(l+1)p^l \geq 0$$

treated in (2.5).

Theorem 2.7 *Suppose given $j \in [0,p^n-1]\backslash(p)$. Let $N(j) := \#\left([0,j]\backslash(p)\right)$. Write $j = \sum_{s\geq 0} a_s p^s$ with $a_s \in [0,p-1]$. The $N(j)$th $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linear elementary divisor of the cyclotomic Dedekind embedding*

$$\begin{array}{rcl} \mathbf{Z}_{(p)}[\zeta_{p^n}] \otimes \mathbf{Z}_{(p)}[\zeta_{p^n}] & \stackrel{\delta_{p^n}}{\hookrightarrow} & \prod_{i\in(\mathbf{Z}/p^n)^*} \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ \zeta_{p^n}^k \otimes \zeta_{p^n}^l & \longmapsto & (\zeta_{p^n}^{k+il})_{i\in(\mathbf{Z}/p^n)^*}, \end{array}$$

where $k, l \in [0, (p-1)p^{n-1}-1]$*, has valuation*

$$-1 + \sum_{s \geq 0} \left(a_s(s+1) - a_{s+1}(s+2)\right) p^s$$

at t*.*

This follows by (1.14 i) using (2.6, 2.4, 2.2).

Remark 2.8 Plesken gives a system of ties that describes the image of the Dedekind embedding case $n = 1$ [P 80, p. 60].

Remark 2.9 If $n = 1$, the valuation at t of the determinant of this embedding δ_p is $-(p-1) + \sum_{a \in [1,p-1]} a = (p-1)(p-2)/2$. If $n \geq 2$, we obtain the valuation at t of the determinant of δ_{p^n} to be

$$\begin{array}{rl}
& -(p-1)p^{n-1} \;+ \displaystyle\sum_{(a_l)_{l\in[0,n-1]} \in [0,p-1]^n,\ a_n = 0,\ a_0 \neq 0} \;\; \sum_{l \in [0,n-1]} (a_l(l+1) - a_{l+1}(l+2))p^l \\
= & -(p-1)p^{n-1} \;+\; p^{n-2} \cdot \displaystyle\sum_{a \in [1,p-1],\ b \in [0,p-1]} (a - 2b) \\
& \;+\; \displaystyle\sum_{l \in [1,n-2]} (p-1)p^{n-3} \cdot \sum_{a,b \in [0,p-1]} (a(l+1) - b(l+2))p^l \\
& \;+\; (p-1)p^{n-2} \cdot \displaystyle\sum_{a \in [0,p-1]} anp^{n-1} \\
= & -(p-1)p^{n-1} \;+\; p^{n-2} \cdot (p-1)(-p^2 + 2p)/2 \\
& \;-\; (p-1)p^{n-2} \cdot p(p-1)/2 \cdot p(p^{n-2} - 1)/(p-1) \\
& \;+\; (p-1)p^{n-2} \cdot p(p-1)/2 \cdot np^{n-1} \\
= & \; p^{2n-2}(p-1)((p-1)n - 1)/2.
\end{array}$$

Hence for $n \geq 1$, we recalculated the valuation at p of the discriminant of $\mathbf{Z}_{(p)}[\zeta_{p^n}]$ over $\mathbf{Z}_{(p)}$ to be $p^{n-1}((p-1)n-1)$. Cf. [N 91, 10.1] (or 5.9).

Example 2.10 Let $p = 3$ and $n = 2$, so that

$$\tau \;=\; (1-\zeta_9,\, 1-\zeta_9^2,\, 1-\zeta_9^4,\, 1-\zeta_9^5,\, 1-\zeta_9^7,\, 1-\zeta_9^8).$$

As elementary divisors of the Dedekind embedding

$$\begin{array}{ccccc}
\mathbf{Z}_{(3)}[\zeta_9] & \otimes & \mathbf{Z}_{(3)}[\zeta_9] & \overset{\delta_9}{\hookrightarrow} & \prod_{j \in (\mathbf{Z}/9)^*} \mathbf{Z}_{(3)}[\zeta_9] \\
\zeta_9^k & \otimes & \zeta_9^l & \longmapsto & (\zeta_9^k \zeta_9^{jl})_{j \in (\mathbf{Z}/9)^*},
\end{array}$$

where $k, l \in [0, 5]$, (2.7) yields

$$(t^0,\, t^1,\, t^4,\, t^5,\, t^8,\, t^9).$$

The determinant of δ_9 has valuation 27 at t (cf. 2.9). By (1.14 iii), a triangular $\mathbf{Z}_{(3)}[\zeta_9]$-linear basis of the image of δ_9 is given by the rows of the matrix

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & -\zeta_9+\zeta_9^2 & -\zeta_9+\zeta_9^3 & -\zeta_9+\zeta_9^4 & -2\zeta_9-\zeta_9^4 & -\zeta_9-\zeta_9^2-\zeta_9^5 \\
0 & 0 & -1-\zeta_9^4-\zeta_9^5 & 1-\zeta_9^2+2\zeta_9^3-2\zeta_9^5 & -1+\zeta_9^2+\zeta_9^3+2\zeta_9^5 & -1-2\zeta_9+\zeta_9^3-\zeta_9^4 \\
0 & 0 & 0 & 1-2\zeta_9-2\zeta_9^2+2\zeta_9^3-\zeta_9^4-\zeta_9^5 & 2+2\zeta_9+2\zeta_9^2+4\zeta_9^3+\zeta_9^5 & -1-\zeta_9+2\zeta_9^2+\zeta_9^3+\zeta_9^4+\zeta_9^5 \\
0 & 0 & 0 & 0 & 3+6\zeta_9+3\zeta_9^3-3\zeta_9^5 & 3\zeta_9+3\zeta_9^2+3\zeta_9^3+3\zeta_9^4+3\zeta_9^5 \\
0 & 0 & 0 & 0 & 0 & 3\zeta_9^2+3\zeta_9^4+3\zeta_9^5
\end{bmatrix}.$$

3 The cyclic Wedderburn embedding

3.1 The q-Pascal method

We diagonalize the Vandermonde matrix that describes the Wedderburn embedding in the $\mathbf{Z}[\zeta_m]$-linear basis $(c_m^0, \ldots, c_m^{m-1})$ of $\mathbf{Z}[\zeta_m]C_m$. More specifically speaking, we multiply with a lower triangular matrix G_q that records the q-Pascal triangle, consisting of Gaußian polynomials, from the right and with its transpose G_q^{t} from the left. The resulting diagonal matrix is of elementary divisor form, regardless whether or not the ground ring $\mathbf{Z}[\zeta_m]$ is a principal ideal domain. We derive the necessary identities firstly in a pedestrian fashion, yielding in particular a formula for G_q^{-1}, and secondly, as a consequence of F. H. Jackson's q-analogue (1910) of L. Saalschütz' theorem (1890).

Notation 3.1 Let $m \geq 1$. The *cyclic Wedderburn embedding* is given by

$$\begin{array}{rcl} \mathbf{Z}[\zeta_m]C_m & \stackrel{\omega_m}{\hookrightarrow} & \prod_{j\in[0,m-1]} \mathbf{Z}[\zeta_m] \\ c_m & \longmapsto & (\zeta_m^j)_{j\in[0,m-1]}. \end{array}$$

3.1.1 Pedestrian

Notation 3.2 (Gaußian polynomials) Consider the field $\mathbf{Q}(q)$ of rational functions in the indeterminate q. Given $i \geq 0$ and $j \in \mathbf{Z}$ and another indeterminate t, we let

$$\begin{array}{rcl} [i] & := & (q^i - 1)/(q-1) \\ [i]! & := & \prod_{k\in[1,i]}[k] \\ \begin{bmatrix} i \\ j \end{bmatrix} & := & \begin{cases} \frac{[i]!}{[j]![i-j]!} & \text{if } j \in [0,i] \\ 0 & \text{if } j \notin [0,i] \end{cases} \\ (t;q)_i & := & \prod_{k\in[0,i-1]}(1-q^k t). \end{array}$$

In particular, $[0]! = 1$. If necessary, we indicate q as $\begin{bmatrix} i \\ j \end{bmatrix}_q := \begin{bmatrix} i \\ j \end{bmatrix}$. Let the *$q$-Pascal matrix* of size $m \times m$ be defined by

$$G_q := \left(\begin{bmatrix} i \\ j \end{bmatrix}\right)_{i,j\in[0,m-1]}.$$

Moreover, we let

$$\begin{array}{rcl} I & := & (\partial_{i,j})_{i,j\in[0,m-1]} \\ V_q & := & (q^{ij})_{i,j\in[0,m-1]} \\ D_q & := & \left(\partial_{i,j}[i]!(q-1)^i q^{\binom{i}{2}}\right)_{i,j\in[0,m-1]}. \end{array}$$

Lemma 3.3 *If $i \geq 1$ and $j \geq 0$, we have*

$$\begin{array}{rcl} \begin{bmatrix} i \\ j \end{bmatrix} & = & \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} i-1 \\ j \end{bmatrix} \\ & = & q^{i-j} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} i-1 \\ j \end{bmatrix}. \end{array}$$

Moreover,

$$(t;q)_i \;=\; \sum_{k\in[0,i]} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} i \\ k \end{bmatrix} t^k.$$

Finally,

$$\begin{bmatrix} i \\ j \end{bmatrix}_{q^{-1}} \;=\; q^{-j(i-j)} \begin{bmatrix} i \\ j \end{bmatrix}_q .$$

Proposition 3.4 (Inversion of the q-Pascal matrix)
We obtain the inverse of G_q to be

$$G_q^{-1} = \left((-1)^{j+k} q^{\binom{j-k}{2}} \begin{bmatrix} j \\ k \end{bmatrix} \right)_{j,k\in[0,m-1]} .$$

Given $0 \le k \le i \le m-1$, we need to show that

$$\sum_{j\in[k,i]} \begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} (-1)^{j+k} q^{\binom{j-k}{2}} \;=\; \partial_{i,k}.$$

We perform an induction on $i-k$. If $i=k$, we obtain $1=1$. If $i=k+1$, we obtain $0=0$. If $i \ge k+2$, we obtain

$$\begin{aligned}
\sum_{j\ge 0} \begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} (-1)^{j+k} q^{\binom{j-k}{2}} \quad & \overset{(3.3)}{=} \quad \sum_{j\ge 0} \begin{bmatrix} i-1 \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} (-1)^{j+k} q^{\binom{j-k}{2}} \\
& + \quad \sum_{j\ge 0} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} j-1 \\ k \end{bmatrix} (-1)^{j+k} q^{\binom{j-k}{2}+(i-j)} \\
& + \quad \sum_{j\ge 0} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} (-1)^{j+k} q^{\binom{j-k}{2}+(i-j)+(j-k)} \\
& \overset{\text{induction}}{=} \quad \sum_{j\ge 0} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} (-1)^{j+k} q^{\binom{j-k}{2}+i-k}.
\end{aligned}$$

The last term vanishes by subinduction on k, since if $k=0$ we obtain

$$\sum_{j\ge 0} \begin{bmatrix} i \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}} \;\overset{(3.3)}{=}\; (1;q)_i \;=\; 0.$$

Lemma 3.5 *We have*

$$V_q (G_q^{-1})^{\mathrm{t}} \;=\; G_q D_q.$$

We need to show that for $i,k \in [0,m-1]$

$$\sum_{j\in[0,k]} q^{ij} (-1)^{j+k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{k-j}{2}} \;=\; q^{\binom{k}{2}} [k]!(q-1)^k \begin{bmatrix} i \\ k \end{bmatrix} .$$

We perform an induction on k, starting in case $k=0$ with $1=1$. If $k \geq 1$, we calculate

$$\begin{aligned}
\sum_{j\in[0,k]} q^{ij}(-1)^{j+k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{k-j}{2}} & \overset{(3.3)}{=} \sum_{j\in[0,k-1]} q^{i(j+1)}(-1)^{j+1+k} \begin{bmatrix} k-1 \\ j \end{bmatrix} q^{\binom{k-j-1}{2}} \\
& + \sum_{j\in[0,k-1]} q^{ij}(-1)^{j+k} \begin{bmatrix} k-1 \\ j \end{bmatrix} q^{\binom{k-j}{2}+j} \\
& \overset{\text{induction}}{=} (q^i - q^{k-1}) q^{\binom{k-1}{2}} [k-1]!(q-1)^{k-1} \begin{bmatrix} i \\ k-1 \end{bmatrix} \\
& = q^{\binom{k}{2}} [k]!(q-1)^k \begin{bmatrix} i \\ k \end{bmatrix}.
\end{aligned}$$

Lemma 3.6 (Fourier inversion) *We have $V_{\zeta_m} V_{\zeta_m^{-1}} = mI$. In particular, reordering the rows of $V_{\zeta_m^{-1}}$ shows that*

$$(\det V_{\zeta_m})^2 = \begin{cases} (-1)^{\frac{m-1}{2}} m^m & \text{if } m \text{ is odd} \\ (-1)^{\frac{m-2}{2}} m^m & \text{if } m \text{ is even.} \end{cases}$$

In fact,

$$\begin{aligned}
\textstyle\sum_{j\in[0,m-1]} \zeta_m^{ij}\zeta_m^{-jk} &= \textstyle\sum_{j\in[0,m-1]} \zeta_m^{(i-k)j} \\
&= m\partial_{i,k}.
\end{aligned}$$

Corollary 3.7 (cf. [N 91, 8.6]) *If m is odd, then $\mathbf{Q}(\zeta_m)$ contains $\sqrt{(-1)^{\frac{m-1}{2}} m}$.*

Proposition 3.8

(i) We have the diagonalization

$$G^{\mathrm{t}}_{\zeta_m^{-1}} V_{\zeta_m} G_{\zeta_m^{-1}} = m(D_{\zeta_m^{-1}})^{-1},$$

the right hand side being a diagonal matrix with $(i+1)$st diagonal entry

$$\frac{m\zeta_m^{(i^2)}}{\prod_{j\in[1,i]}(1-\zeta_m^j)},$$

where $i \in [0, m-1]$.

(ii) We have

$$\begin{aligned}
& (\mathbf{Z}[\zeta_m]C_m)\omega_m \\
= & \left\{ (y_j)_{j\in[0,m-1]} \;\middle|\; \sum_{j\in[i,m-1]} y_j \begin{bmatrix} j \\ i \end{bmatrix}_{\zeta_m^{-1}} \in \mathbf{Z}[\zeta_m] \cdot \frac{m}{\prod_{j\in[1,i]}(1-\zeta_m^j)} \text{ for } i \in [0, m-1] \right\} \\
\subseteq & \textstyle\prod_{j\in[0,m-1]} \mathbf{Z}[\zeta_m].
\end{aligned}$$

(iii) A $\mathbf{Z}[\zeta_m]$-linear basis of $(\mathbf{Z}[\zeta_m]C_m)\omega_m$ is given by the following tuple of elements of $\prod_{j\in[0,m-1]}\mathbf{Z}[\zeta_m]$.

$$\left(\left((-1)^k\zeta_m^{\binom{k}{2}}\frac{m}{\prod_{l\in[1,j]}(1-\zeta_m^l)}\begin{bmatrix} j\\ k\end{bmatrix}_{\zeta_m}\right)_{k\in[0,m-1]}\right)_{j\in[0,m-1]}$$

Ad (i). This formula follows from (3.5, 3.6).

Ad (ii). A vector $y=(y_j)_{j\in[0,m-1]}$ is contained in the image of ω_m if and only if $yV_{\zeta_m}^{-1}$ has entries in $\mathbf{Z}[\zeta_m]$. But $V_{\zeta_m}^{-1}(G^{\mathrm{t}}_{\zeta_m^{-1}})^{-1}=G_{\zeta_m^{-1}}D_{\zeta_m^{-1}}m^{-1}$, so this condition translates into $yG_{\zeta_m^{-1}}D_{\zeta_m^{-1}}m^{-1}$ to have entries in $\mathbf{Z}[\zeta_m]$, which is the defining condition given above.

Ad (iii). This follows by $G^{\mathrm{t}}_{\zeta_m^{-1}}V_{\zeta_m}=m(D_{\zeta_m^{-1}})^{-1}(G_{\zeta_m^{-1}})^{-1}$ using (3.4, 3.3).

Remark 3.9 Because the derivative of

$$F(X)\overset{1.}{=}X^m-1\overset{2.}{=}\prod_{j\in[0,m-1]}(X-\zeta_m^j)$$

evaluated at 1 yields

$$F'(1)\overset{1.}{=}m\overset{2.}{=}\prod_{j\in[1,m-1]}(1-\zeta_m^j),$$

the mth diagonal entry in (3.8 i) equals ζ_m. This reproves in particular that all diagonal entries of (3.8 i) are contained in $\mathbf{Z}[\zeta_m]$.

Remark 3.10 Using (3.9), for $n\geq 1$, p prime, $m=p^n$, the valuation at $t=1-\zeta_{p^n}$ of the $(i+1)$th elementary divisor of ω_{p^n}, $i\in[0,p^n-1]$, is given by

$$\begin{aligned} v_t\left(\frac{p^n}{\prod_{j\in[1,p^n-1-i]}(1-\zeta_{p^n}^j)}\right) &= \textstyle\sum_{j\in[1,i]}v_t\left(1-\zeta_{p^n}^{-j}\right)\\ &\overset{(2.2)}{=} \textstyle\sum_{j\in[1,i]}j[p], \end{aligned}$$

in accordance with (3.15) below in view of (2.3).

3.1.2 Invoking q-Saalschütz

Following J. Stokman, we cite the

Theorem 3.11 (Saalschütz, Jackson **[A 76, 3.3.12]** ([1])) *Given $n\geq 0$, and further indeterminates a, b, c, we have*

$$\sum_{m\geq 0}\frac{(a;q)_m(b;q)_m(q^{-n};q)_m}{(q;q)_m(c;q)_m(abq^{1-n}c^{-1};q)_m}\cdot q^m=\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n}.$$

and specialize to the

Corollary 3.12 ($\Longleftrightarrow$ 3.5) *Given $m \geq 1$, we have*

$$V_q \;=\; G_q D_q (G_q)^{\mathrm{t}}.$$

Given $i, k \in [0, m-1]$, we need to show that

$$q^{ik} = \sum_{j\in[0,m-1]} \begin{bmatrix} i \\ j \end{bmatrix} \cdot [j]!(q-1)^j q^{\binom{j}{2}} \cdot \begin{bmatrix} k \\ j \end{bmatrix}.$$

Substituting q by q^{-1}, n by k and specializing c and then b to zero (after rewriting the right hand side), and specializing a to q^i, (3.11) becomes

$$\begin{aligned} q^{ik} \;&=\; \textstyle\sum_{j\geq 0} \dfrac{(q^i;q^{-1})_j (q^k;q^{-1})_j}{(q^{-1};q^{-1})_j} \cdot q^{-j} \\ &=\; \textstyle\sum_{j\geq 0} \dfrac{(1-q^i)\cdots(1-q^{i-j+1})\cdot(1-q^k)\cdots(1-q^{k-j+1})}{(1-q^{-1})\cdots(1-q^{-j})} \cdot q^{-j} \\ &=\; \textstyle\sum_{j\geq 0} \dfrac{(q^i-1)\cdots(q^{i-j+1}-1)}{(q^1-1)\cdots(q^j-1)} \cdot \dfrac{(q^k-1)\cdots(q^{k-j+1}-1)}{(q^1-1)\cdots(q^j-1)} \cdot q^{\binom{j}{2}}[j]!(q-1)^j \\ &=\; \textstyle\sum_{j\geq 0} \begin{bmatrix} i \\ j \end{bmatrix} \cdot \begin{bmatrix} k \\ j \end{bmatrix} \cdot q^{\binom{j}{2}}[j]!(q-1)^j. \end{aligned}$$

3.2 The general Vandermonde method

To compare, and to recalculate elementary divisors, we apply the method of section 1.2 to the localized cyclic Wedderburn embedding.

Setup 3.13 Let p be a prime, let $n \geq 1$. In the notation of (1.13), we place ourselves in the situation $S = \mathbf{Z}_{(p)}$, $s = p$, $T = \mathbf{Z}_{(p)}[\zeta_{p^n}]$ and $t = \vartheta = 1 - \zeta_{p^n}$. We consider the localized cyclic Wedderburn embedding

$$\begin{array}{rcl} \mathbf{Z}_{(p)}[\zeta_{p^n}]C_{p^n} & \xrightarrow[\sim]{\omega_{p^n}} & \prod_{j\in\mathbf{Z}/p^n} \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ c_{p^n}^i & \longrightarrow & (\zeta_{p^n}^{ij})_{j\in\mathbf{Z}/p^n}, \end{array}$$

where $i \in [0, p^n - 1]$. In the notation of (1.1), and with respect to the $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linear basis $((1-c_{p^n})^i)_{i\in[0,p^n-1]}$ of $\mathbf{Z}_{(p)}[\zeta_{p^n}]C_{p^n}$ and the standard basis on the right hand side, this embedding is given by the Vandermonde matrix V_τ for $\tau := (1-\zeta_{p^n}^j)_{j\in[0,p^n-1]}$.

Lemma 3.14 *The tuple $\tau := (1-\zeta_{p^n}^j)_{j\in[0,p^n-1]}$ is minimally ordered.*

Using (2.2), we need to see that

$$\sum_{i\in[0,j-1]} (j-i)[p] \leq \sum_{i\in[0,j-1]} (k-i)[p]$$

[1] In the left hand side summand of [A 76, 3.3.14], a factor $(ab/c)^{N-n}$ has been forgotten.

for $j \in [0, p^n - 1]$ and $k \in [j+1, p^n - 1]$. Write $j = \sum_{l \geq 0} a_l p^l$, $k = \sum_{l \geq 0} a'_l p^l$, $k - j = \sum_{l \geq 0} a''_l p^l$, where $a_l, a'_l, a''_l \in [0, p-1]$. By (2.3), we need to show that

$$\sum_{l \geq 0} \Big((a'_l - a''_l - a_l) - (a'_{l+1} - a''_{l+1} - a_{l+1}) \Big) (l+1) p^l \geq 0.$$

This follows from (2.5).

Proposition 3.15 *Suppose given $j \in [0, p^n - 1]$. Write $j = \sum_{k \geq 0} a_k p^k$, $a_k \in [0, p-1]$. The $(j+1)$st elementary divisor of the localized cyclic Wedderburn embedding*

$$\begin{array}{rcl} \mathbf{Z}_{(p)}[\zeta_{p^n}]C_{p^n} & \overset{\omega_{p^n}}{\hookrightarrow} & \prod_{j \in \mathbf{Z}/p^n} \mathbf{Z}_{(p)}[\zeta_{p^n}] \\ c_{p^n}^i & \longmapsto & (\zeta_{p^n}^{ij})_{j \in \mathbf{Z}/p^n} \end{array}$$

where $i \in [0, p^n - 1]$, has valuation

$$\sum_{k \geq 0} (a_k - a_{k+1})(k+1) p^k$$

at t.

This follows by (1.7, 1.12), using (3.14, 2.3), or by (3.10), using (2.3).

Remark 3.16 By (3.6), the square of the determinant of ω_{p^n} has valuation np^n at p. Alternatively, by (3.15), we obtain the valuation at t of the determinant of this embedding to be

$$\begin{array}{l} \displaystyle\sum_{(a_l)_{l \in [0,n-1]} \in [0,p-1]^n,\ a_n = 0} \ \sum_{l \in [0,n-1]} (a_l - a_{l+1})(l+1) p^l \\ \displaystyle = \left(\sum_{l \in [0,n-2]} p^{n-2} \cdot \sum_{a,b \in [0,p-1]} (a-b)(l+1) p^l \right) + \left(p^{n-1} \cdot \sum_{a \in [0,p-1]} anp^{n-1} \right) \\ \displaystyle = p^{n-1} \tfrac{p(p-1)}{2} np^{n-1}. \end{array}$$

3.3 The Pascal method

3.3.1 First order Pascal ties

There are some obvious ties, i.e. congruences of tuple entries, that are necessary for elements of $\prod_{i \in [0,p^n-1]} \mathbf{Z}[\zeta_{p^n}]$ to lie in $(\mathbf{Z}[\zeta_{p^n}]C_{p^n})\omega_{p^n}$. If $n = 1$, they are already sufficient, yielding a manageable basis of $(\mathbf{Z}[\zeta_p]C_p)\omega_p$. In general, they describe an intermediate ring between $(\mathbf{Z}[\zeta_{p^n}]C_{p^n})\omega_{p^n}$ and $\prod_{i \in [0,p^n-1]} \mathbf{Z}[\zeta_{p^n}]$.

Notation 3.17 Suppose given $s \geq 0$, a polynomial $f(X) := \sum_{i \geq 0} a_i X^i \in \mathbf{C}[X]$ and a tuple $(y_j)_{j \in [0,s]}$, $y_j \in \mathbf{C}$. We define the *evaluation of f at $(y_j)_{j \in [0,s]}$* to be

$$\left(\sum_{i \geq 0} a_i X^i \right) \Big[(y_j)_{j \in [0,s]} \Big] := f \Big[(y_j)_{j \in [0,s]} \Big] := \sum_{j \in [0,s]} a_j y_j.$$

We note the difference between the polynomial power $f^i\left[(y_j)_{j\in[0,s]}\right]$ (power of f, taken in $\mathbf{C}[X]$, evaluated) and the ordinary power $f\left[(y_j)_{j\in[0,s]}\right]^i$ (power of the evaluation of f, taken in $\mathbf{C}$), $i \geq 0$.

Let $m \geq 1$, let $t = 1 - \zeta_m$. We consider the $\mathbf{Z}[\zeta_m]$-submodule $W_m^{(1)}$ of

$$\boxed{W_m^{(0)} := \prod_{j\in[0,m-1]} \mathbf{Z}[\zeta_m]}$$

defined by the *first order Pascal ties*

$$\boxed{\begin{array}{rcl} W_m^{(1)} & := & \left\{(y_j)_{j\in[0,m-1]} \;\middle|\; \left((1-X)^i\right)\left[(y_j)_{j\in[0,m-1]}\right] \equiv_{t^i} 0 \text{ for all } i\in[0,m-1]\right\} \\ & = & \left\{(y_j)_{j\in[0,m-1]} \;\middle|\; \sum_{j\in[0,i]}(-1)^j\binom{i}{j} y_j \equiv_{t^i} 0 \text{ for all } i\in[0,m-1]\right\} \\ & \subseteq & W_m^{(0)}. \end{array}}$$

Lemma 3.18

(i) *The image* $(\mathbf{Z}[\zeta_m]C_m)\omega_m$ *of the Wedderburn embedding (3.1) is contained in* $W_m^{(1)}$,

$$\mathbf{Z}[\zeta_m]C_m \xrightarrow{\omega_m} W_m^{(1)} \hookrightarrow W_m^{(0)}.$$

(ii) *A* $\mathbf{Z}[\zeta_m]$*-linear basis of* $W_m^{(1)}$ *is given by*

$$(\xi_{m,i})_{i\in[0,m-1]} := \left(\left((-1)^i t^i \binom{j}{i}\right)_{j\in[0,m-1]}\right)_{i\in[0,m-1]}.$$

(iii) *The* $(i+1)$*st elementary divisor over* $\mathbf{Z}[\zeta_m]$ *of the embedding* $W_m^{(1)} \subseteq W_m^{(0)}$ *is given by* t^i, $i \in [0, m-1]$. *(In particular,* $\mathbf{Z}[\zeta_m]$ *not being a principal ideal domain in general, there exist bases with respect to which the matrix that describes this embedding takes diagonal shape.)*

(iv) *The* $\mathbf{Z}[\zeta_m]$*-linear determinant of the embedding* $W_m^{(1)} \subseteq W_m^{(0)}$ *is* $t^{m(m-1)/2}$.

Ad (i). We have $\left((1-X)^i\right)\left[(\zeta_m^{lj})_{j\in[0,m-1]}\right] = (1-\zeta_m^l)^i \equiv_{t^i} 0$ for all $i \in [0, m-1]$ and all $l \in [0, m-1]$.

Ad (ii). Inverting the matrix $A := \left((-1)^i t^i \binom{j}{i}\right)_{i\in[0,m-1],\, j\in[0,m-1]}$ arising from the tuple of elements $(\xi_{m,j})_{j\in[0,m-1]}$, we obtain $A^{-1} := \left((-1)^i t^{-j} \binom{j}{i}\right)_{i\in[0,m-1],\, j\in[0,m-1]}$. An element y is contained in the $\mathbf{Z}[\zeta_m]$-linear span of our tuple if and only if, y considered as a row vector, yA^{-1} is entrywise contained in $\mathbf{Z}[\zeta_m]$, i.e. if and only if $y \in W_m^{(1)}$.

In particular, we have the

Proposition 3.19 *Let p be a prime. We have a factorization of the Wedderburn embedding (3.1) into*

$$\mathbf{Z}[\zeta_p]C_p \xrightarrow[\sim]{\omega_p} W_p^{(1)} \subseteq W_p^{(0)} = \prod_{j\in\mathbf{Z}/p} \mathbf{Z}[\zeta_p].$$

The factorization follows by (3.18 i). The isomorphism follows by comparison of (3.18 iv) with (3.6), both yielding the valuation at t of the determinant of the respective embedding to be $p(p-1)/2$, and zero elsewhere. We remark that the elementary divisors resulting from (3.18 iii) are in accordance with (3.15).

Remark 3.20 (coefficient criterion) *The tuple $(y_i)_{i\in[0,p]} \in W_p^{(0)}$ is contained in $W_p^{(1)}$ if and only if, writing $y_i =: \sum_{j\in[0,p-2]} y_{i,j}\zeta^j$, $y_{i,j} \in \mathbf{Z}$,*

$$\sum_{i\in[0,p-1],\ j\in[0,p-2]} (-1)^i \tbinom{u}{i} \tbinom{j}{v} y_{i,j} \equiv_p 0$$

holds for all $0 \leq v < u \leq p-1$.

An element $\sum_{j\in[0,p-2]} x_j\zeta_p^j = \sum_{v\in[0,p-2]} \left(\sum_{j\in[0,p-2]} x_j \tbinom{j}{v}\right)(\zeta_p - 1)^v$, $x_j \in \mathbf{Z}$, vanishes modulo t^u for $u \in [0, p-1]$ if and only if $\sum_{j\in[0,p-2]} x_j \tbinom{j}{v} \equiv_p 0$ for all $v \in [0, u-1]$. It remains to plug in $x_j = \sum_{i\in[0,p-1]}(-1)^i \tbinom{u}{i} y_{i,j}$.

Lemma 3.21 *Given $0 \leq j \leq i$, we have*

$$\xi_{m,j}\xi_{m,i} = \sum_{k\in[0,j]} \binom{j}{k}\binom{i+k}{j}(-1)^{j-k}t^{j-k}\xi_{m,i+k},$$

where we let $\xi_{m,l} := 0$ for $l \geq m$. In particular, $W_m^{(1)}$ is a subring of $W_m^{(0)}$.

We need to see that for $l \in [0, m-1]$

$$(-1)^j t^j \binom{l}{j} \cdot (-1)^i t^i \binom{l}{i} = \sum_{k\in[0,j]} \binom{j}{k}\binom{i+k}{j}(-1)^{j-k}t^{j-k} \cdot (-1)^{i+k}t^{i+k}\binom{l}{i+k}.$$

We may assume $i \leq l$ and reformulate to

$$\binom{l}{i} = \sum_{k\geq 0} \binom{j}{k}\binom{l-j}{l-i-k},$$

which now follows from a comparison of coefficients in $(1+T)^l = (1+T)^j(1+T)^{l-j}$ at T^{l-i}.

Remark 3.22 *Let p be a prime, let $n \geq 1$, let $m = p^n$ (in particular, $t = 1 - \zeta_{p^n}$). We dispose of ring automorphisms*

$$\begin{array}{rcl} \mathbf{Z}[\zeta_{p^n}]C_{p^n} & \xrightarrow[\sim]{\alpha_{p^n}} & \mathbf{Z}[\zeta_{p^n}]C_{p^n} \\ c_{p^n} & \longrightarrow & \zeta_{p^n}c_{p^n}, \end{array}$$

and

$$\begin{array}{rcl} W_{p^n}^{(0)} & \xrightarrow[\sim]{\alpha_{p^n}^{(0)}} & W_{p^n}^{(0)} \\ (y_j)_{j\in\mathbf{Z}/p^n} & \longrightarrow & (y_{j+1})_{j\in\mathbf{Z}/p^n} \end{array}$$

satisfying $\omega_{p^n}\alpha^{(0)}_{p^n} = \alpha_{p^n}\omega_{p^n}$. *Moreover,* $\alpha^{(0)}_{p^n}$ *restricts to an automorphism* $\alpha^{(1)}_{p^n}$ *of* $W^{(1)}_{p^n}$, *with operation given by*

$$\xi_{p^n,j}\alpha^{(1)}_{p^n} = \xi_{p^n,j} - t\xi_{p^n,j-1} + (-1)^{p-j}t^{j-(p^n-1)}\binom{p^n}{j}\xi_{p^n,p^n-1}$$

for $j \in [1, p^n - 1]$, *and by* $\xi_{p^n,0}\alpha^{(1)}_{p^n} = \xi_{p^n,0} (= 1_{W^{(1)}_{p^n}})$.

For $j \in [1, p^n - 1]$ and $i \in [0, p^n - 1]$, we have

$$\left((-1)^j t^j \tbinom{i}{j}\right) - t\left((-1)^{j-1}t^{j-1}\tbinom{i}{j-1}\right) = (-1)^j t^j \tbinom{i+1}{j},$$

which we compare for $i = p^n - 1$ with

$$\xi_{p^n,p^n-1} = (0, \ldots, 0, (-1)^{p^n-1}t^{p^n-1}),$$

whence the formula describing the operation of $\alpha^{(1)}_{p^n}$. The valuation at t of the third coefficient therein amounts to $j + (n - v_p(j))(p-1)p^{n-1} - (p^n - 1)$. In case $n - v_p(j) \geq 2$, we obtain

$$j + (n - v_p(j))(p-1)p^{n-1} \geq 2(p-1)p^{n-1} \geq p^n - 1.$$

If $n - v_p(j) = 1$, we obtain

$$j + (n - v_p(j))(p-1)p^{n-1} \geq p^{n-1} + (p-1)p^{n-1} \geq p^n - 1.$$

Hence $\alpha^{(0)}_{p^n}$ restricts to an automorphism $\alpha^{(1)}_{p^n}$.

Lemma 3.23 *The factorization of the Wedderburn embedding maps*

$$\begin{array}{rcl} \mathbf{Z}[\zeta_m]C_m & \xrightarrow{\omega_m} & W^{(1)}_m \\ c^i_m & \longrightarrow & \sum_{k\in[0,m-1]}\left(\frac{1-\zeta^i_m}{t}\right)^k \xi_{m,k} \end{array}$$

for $i \in [0, m-1]$, *where we let* $0^0 = 1$.

In fact,

$$\begin{aligned} \textstyle\sum_{k\in[0,m-1]}\left(\frac{1-\zeta^i_m}{t}\right)^k \xi_{m,k} &= \left(\textstyle\sum_{k\in[0,m-1]}\left(\frac{1-\zeta^i_m}{t}\right)^k (-1)^k t^k \tbinom{j}{k}\right)_{j\in[0,m-1]} \\ &= \left((1-(1-\zeta^i_m))^j\right)_{j\in[0,m-1]} \\ &= c^i_m\omega_m, \end{aligned}$$

which remains true for $i = 0$.

3.3.2 Second order Pascal ties

In this appendix to subsection 3.3.1 we shall indicate a method of how to continue the approach via Pascal ties.

Let $n \geq 2$, let p be a prime and let $t = 1 - \zeta_{p^n}$.

Lemma 3.24 *Let* $s \geq 1$. *For any* $i \geq 1$, *we have in* $\mathbf{Q}[X]$

$$\sum_{k\in[1,s]} \frac{1}{k}\left((1-(1-X)^i)^k - iX^k\right) \in (X^{s+1}).$$

This expression being contained in (X), it suffices to prove that its derivative is contained in (X^s). In fact,

$$\begin{aligned} & \tfrac{d}{dX} \sum_{k\in[1,s]} \tfrac{1}{k}\left((1-(1-X)^i)^k - iX^k\right) \\ = & \sum_{k\in[1,s]} \left((1-(1-X)^i)^{k-1} i(1-X)^{i-1} - iX^{k-1}\right) \\ = & \; i\left(\frac{1-(1-(1-X)^i)^s}{1-(1-(1-X)^i)}(1-X)^{i-1} - \frac{1-X^s}{1-X}\right) \\ = & \; \frac{i}{1-X}\left(-(1-(1-X)^i)^s + X^s\right). \end{aligned}$$

Lemma 3.25 *For this lemma, we allow $n \geq 1$. Let*

$$f_{p^n}(X) := \left(\sum_{k\in[1,p-1]} \frac{t^{k-1}}{k}\right) X^p - \sum_{k\in[1,p-1]} \frac{t^{k-1}}{k} X^k \in \mathbf{Z}_{(p)}[\zeta_{p^n}][X].$$

For any $i \in [0, p^n - 1]$, we have

$$f_{p^n}\left(\frac{1-\zeta_{p^n}^i}{t}\right) \equiv_{t^{p-1}} 0.$$

We also write shorthand

$$\gamma := \sum_{k\in[1,p-1]} \frac{t^{k-1}}{k}.$$

In fact,

$$\begin{array}{ll} & \left(\sum_{k\in[1,p-1]} \frac{t^{k-1}}{k}\right)\left(\frac{1-(1-t)^i}{t}\right)^p - \sum_{k\in[1,p-1]} \frac{t^{k-1}}{k}\left(\frac{1-(1-t)^i}{t}\right)^k \\ = & \frac{1}{t}\sum_{k\in[1,p-1]} \frac{1}{k}\left(t^k\left(\frac{1-(1-t)^i}{t}\right)^p - \left(1-(1-t)^i\right)^k\right) \\ \equiv_p & \frac{1}{t}\sum_{k\in[1,p-1]} \frac{1}{k}\left(t^k \frac{1-(1-t^p)^i}{t^p} - \left(1-(1-t)^i\right)^k\right) \\ \equiv_{t^p} & \frac{1}{t}\sum_{k\in[1,p-1]} \frac{1}{k}\left(it^k - \left(1-(1-t)^i\right)^k\right) \\ \overset{(3.24)}{\equiv}_{t^{p-1}} & 0. \end{array}$$

We consider the $\mathbf{Z}[\zeta_{p^n}]$-submodule $W_{p^n}^{(2)}$ of $W_{p^n}^{(1)}$ defined by the *second order Pascal ties*

$$\boxed{\begin{array}{rl} W_{p^n}^{(2)} := & \left\{ \sum_{k\in[0,p^n-1]} z_k \xi_{p^n,k} \;\middle|\; \begin{array}{l} z_k \in \mathbf{Z}[\zeta_{p^n}],\ \left(X^i f_{p^n}^j(X)\right)\left[(z_k)_{k\in[0,p^n-1]}\right] \equiv_{t^{j(p-1)}} 0 \\ \text{for all } i \in [0,p-1] \text{ and all } j \in [0,p^{n-1}-1] \end{array} \right\} \\ \subseteq & W_{p^n}^{(1)}, \end{array}}$$

where the congruence is to be read in $\mathbf{Z}_{(p)}[\zeta_{p^n}]$.

Lemma 3.26

(i) The image $(\mathbf{Z}[\zeta_{p^n}]C_{p^n})\omega_{p^n}$ of the Wedderburn embedding (3.1) is contained in $W_{p^n}^{(2)}$,

$$\mathbf{Z}[\zeta_{p^n}]C_{p^n} \xrightarrow{\omega_{p^n}} W_{p^n}^{(2)} \hookrightarrow W_{p^n}^{(1)} \hookrightarrow W_{p^n}^{(0)}.$$

(ii) The $(i + jp + 1)$th elementary divisor over $\mathbf{Z}_{(p)}[\zeta_{p^n}]$ of the embedding $(W_{p^n}^{(2)})_{(p)} \subseteq (W_{p^n}^{(1)})_{(p)}$, $i \in [0, p-1]$, $j \in [0, p^{n-1}-1]$, is given by $t^{j(p-1)}$. Moreover, the $(i + jp + 1)$th elementary divisor of the embedding $(W_{p^n}^{(2)})_{(p)} \subseteq (W_{p^n}^{(0)})_{(p)}$ is given by $t^{(i+jp)+j(p-1)}$.

(iii) The valuation of the $\mathbf{Z}[\zeta_{p^n}]$*-linear determinant of the embedding* $W^{(2)}_{p^n} \subseteq W^{(1)}_{p^n}$ *is given by* $p^n(p^{n-1}-1)(p-1)/2$ *at* t *and by zero elsewhere.*

Ad (i). We have to take care of the coefficients calculated in (3.23). For $i \in [0, p-1]$, $j \in [0, p^{n-1}-1]$ and $h \in [0, p^n - 1]$ we obtain

$$\left(X^i f^j_{p^n}(X)\right)\left[\left(\left(\frac{1-\zeta^h_{p^n}}{t}\right)^k\right)_{k\in[0,p^n-1]}\right] = \left(\frac{1-\zeta^h_{p^n}}{t}\right)^i f^j_{p^n}\left(\frac{1-\zeta^h_{p^n}}{t}\right) \overset{(3.25)}{\equiv}_{t^{j(p-1)}} 0.$$

Question 3.27 We do not know whether $W^{(2)}_{p^n}$ is a subring of $W^{(1)}_{p^n}$.

Specializing to $n = 2$, we obtain the

Proposition 3.28 *We have a factorization of the Wedderburn embedding into*

$$\mathbf{Z}[\zeta_{p^2}]C_{p^2} \xrightarrow[\sim]{\omega_{p^2}} W^{(2)}_{p^2} \subseteq W^{(1)}_{p^2} \subseteq W^{(0)}_{p^2} = \prod_{j\in\mathbf{Z}/p^2} \mathbf{Z}[\zeta_{p^2}].$$

The factorization follows by (3.26 i). The isomorphism follows by comparison of (3.26 iii) and (3.18 iv) with (3.6), both yielding the valuation at t of the determinant of the respective embedding to be $p^3(p-1)$, and zero elsewhere. We remark that in this case, the elementary divisors resulting from (3.26 ii) are in accordance with (3.15).

Example 3.29 Consider the case $p = 3$, $n = 2$, thus $t = 1 - \zeta_9$, $\gamma = 1 + t/2$. An element $\sum_{k\in[0,8]} z_k \xi_{9,k}$ is contained in $W^{(2)}_9$ if and only if, $z = (z_k)_{k\in[0,8]}$ considered as a row vector, it multiplies with

$$\left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -t/2 & -1 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & \gamma & -t/2 & -1 & t & 1 & 0 \\
0 & 0 & 0 & 0 & \gamma & -t/2 & -2\gamma+t^2/4 & t & 1 \\
0 & 0 & 0 & 0 & 0 & \gamma & -t\gamma & -2\gamma+t^2/4 & t \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \gamma^2 & -t\gamma & -2\gamma+t^2/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma^2 & -t\gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma^2
\end{array}\right]
\left[\begin{array}{ccc|ccc|ccc}
t^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t^0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & t^{-2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t^{-2} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & t^{-4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-4}
\end{array}\right]$$

to a vector entrywise contained in $\mathbf{Z}_{(3)}[\zeta_9]$.

3.4 Comparison of methods in an example

Example 3.30 Consider the case $m = 5$, in which the cyclic Wedderburn embedding becomes

$$\begin{array}{rcl}
\mathbf{Z}[\zeta_5]C_5 & \overset{\omega_5}{\hookrightarrow} & \prod_{j\in[0,4]} \mathbf{Z}[\zeta_5] \\
c_5 & \longmapsto & (\zeta_5^j)_{j\in[0,4]}.
\end{array}$$

Let $t = 1 - \zeta_5$.

(i) The q-Pascal method (3.8 iii) yields the $\mathbf{Z}[\zeta_5]$-linear basis of $(\mathbf{Z}[\zeta_5]C_5)\omega_5$ given by the rows of

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 4+3\zeta_5+2\zeta_5^2+\zeta_5^3 & -4-3\zeta_5-2\zeta_5^2-\zeta_5^3 & 0 & 0 & 0 \\ 2+\zeta_5+2\zeta_5^2 & -1-\zeta_5+2\zeta_5^3 & 2\zeta_5+\zeta_5^2+2\zeta_5^3 & 0 & 0 \\ 2+\zeta_5+\zeta_5^2+\zeta_5^3 & -2-\zeta_5^2-2\zeta_5^3 & 3+3\zeta_5+2\zeta_5^2+2\zeta_5^3 & \zeta_5^2-\zeta_5^3 & 0 \\ 1 & -1+\zeta_5+\zeta_5^3 & -2-\zeta_5-2\zeta_5^2+\zeta_5^3 & -3\zeta_5^2-\zeta_5^3 & \zeta_5 \end{bmatrix}.$$

(ii) The general Vandermonde method (1.6) yields the $\mathbf{Z}[\zeta_5]$-linear basis of $(\mathbf{Z}[\zeta_5]C_5)\omega_5$ given by the rows of

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & (\zeta_5-1) & (\zeta_5^2-1) & (\zeta_5^3-1) & (\zeta_5^4-1) \\ 0 & 0 & (\zeta_5^2-1)(\zeta_5^2-\zeta_5) & (\zeta_5^3-1)(\zeta_5^3-\zeta_5) & (\zeta_5^4-1)(\zeta_5^4-\zeta_5) \\ 0 & 0 & 0 & (\zeta_5^3-1)(\zeta_5^3-\zeta_5)(\zeta_5^3-\zeta_5^2) & (\zeta_5^4-1)(\zeta_5^4-\zeta_5)(\zeta_5^4-\zeta_5^2) \\ 0 & 0 & 0 & 0 & (\zeta_5^4-1)(\zeta_5^4-\zeta_5)(\zeta_5^4-\zeta_5^2)(\zeta_5^4-\zeta_5^3) \end{bmatrix}.$$

(iii) The Pascal method (3.18 ii, 3.19) yields the $\mathbf{Z}[\zeta_5]$-linear basis of $(\mathbf{Z}[\zeta_5]C_5)\omega_5$ given by the rows of

$$\begin{array}{c} \xi_{5,0} \\ \xi_{5,1} \\ \xi_{5,2} \\ \xi_{5,3} \\ \xi_{5,4} \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -t & -2t & -3t & -4t \\ 0 & 0 & t^2 & 3t^2 & 6t^2 \\ 0 & 0 & 0 & -t^3 & -4t^3 \\ 0 & 0 & 0 & 0 & t^4 \end{bmatrix}.$$

But this method works only for m prime (and for $m = p^2$, but less simply).

4 The radical series of $(W_{p^n}^{(1)})_{(p)}$

Let T be a discrete valuation ring with maximal ideal generated by t and residue field $k = T/tT$. Let Λ be a subalgebra over T of a direct product of m copies of T such that its embedding into this product has torsion cokernel. The dimension over k of the radical layers $\mathfrak{r}^i\Lambda/\mathfrak{r}^{i+1}\Lambda$ stabilizes for large i at m [Kü 99, E.2.3]. On the other hand, the surjection

$$\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda \otimes_k \mathfrak{r}^i\Lambda/\mathfrak{r}^{i+1}\Lambda \longrightarrow \mathfrak{r}^{i+1}\Lambda/\mathfrak{r}^{i+2}\Lambda$$

induced by multiplication yields the bound

$$(\dim_k \mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda)(\dim_k \mathfrak{r}^i\Lambda/\mathfrak{r}^{i+1}\Lambda) \geq \dim_k(\mathfrak{r}^{i+1}\Lambda/\mathfrak{r}^{i+2}\Lambda).$$

The question is the behaviour of this sequence of dimensions $\dim_k(\mathfrak{r}^i\Lambda/\mathfrak{r}^{i+1}\Lambda)$. The ring $W_{p^n}^{(1)}$ introduced in (3.17), localized at (p), yields some example material.

Notation 4.1 Let p be a prime, let $n \geq 1$ and write $t = 1 - \zeta_{p^n}$. For $i \in \mathbf{Z}$, denote $\underline{i} := \max(0, i)$. Given an integer $i \geq 0$, we write it as $i = \sum_{j \geq 0} a_j p^j$ with $a_j \in [0, p-1]$ and denote its p-adic Quersumme by $q_p(i) := \sum_{j \geq 0} a_j$.

For $a, b, m \geq 0$, we let

$$\left(\sum_{i \in [0,m]} T^i \right)^a =: \sum_{b \geq 0} \binom{a}{b}_m T^b \in \mathbf{Z}[T].$$

That is, $\binom{a}{b}_m$ denotes the number of of choices of b balls out of a different balls with each ball chosen at most m times, disregarding the order in which we choose.

In particular, $\binom{a}{b}_1 = \binom{a}{b}$. The sequence $(\binom{a}{b}_m)_{m \geq 0}$ becomes stationary for $m \geq b$ at the value $\binom{a+b-1}{b}$. We have the recursion formula

$$\binom{a+1}{b}_m = \sum_{j \in [0,m]} \binom{a}{b-j}_m,$$

where we let $\binom{a}{c}_m := 0$ for $c < 0$. Furthermore, plugging in $T = 1$ in the definition yields $\sum_{b \in [0,am]} \binom{a}{b}_m = (m+1)^a$.

Proposition 4.2

(i) The $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-order $(W^{(1)}_{p^n})_{(p)}$ is a local ring, with simple module $\mathbf{F}_p$ acted upon identically by $\xi_{p^n,0}$ and trivially by $\xi_{p^n,i}$ for $i \in [1, p-1]$ (cf. 3.18 ii). I.e.

$$\mathfrak{r}(W^{(1)}_{p^n})_{(p)} = (W^{(1)}_{p^n})_{(p)} \cap t(W^{(0)}_{p^n})_{(p)}$$

(ii) For $i \geq 0$, the ideal $\mathfrak{r}^i(W^{(1)}_{p^n})_{(p)}$ is $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-linearly generated by the tuple

$$\left(t^{\underline{i - q_p(j)}} \xi_{p^n,j} \right)_{j \in [0, p^n - 1]}.$$

In particular, for $i \geq n(p-1)$ we have $\mathfrak{r}^{i+1}(W^{(1)}_{p^n})_{(p)} = t\mathfrak{r}^i(W^{(1)}_{p^n})_{(p)}$ (cf. [Kü 99, E.2.3]).

(iii) On the quantitative side, we obtain for $i \geq 0$

$$l_{p^n,i} := \dim_{\mathbf{F}_p} \mathfrak{r}^i(W^{(1)}_{p^n})_{(p)} / \mathfrak{r}^{i+1}(W^{(1)}_{p^n})_{(p)} = \sum_{j \in [0,i]} \binom{n}{j}_{p-1}.$$

Ad (i). On the one hand, the quotient of the respective $\mathbf{Z}_{(p)}[\zeta_{p^n}]$-order modulo the ideal claimed to be its radical is a simple module. On the other hand, this ideal is nilpotent modulo t.

Ad (ii). We abbreviate $\mathfrak{r}^i := \mathfrak{r}^i(W^{(1)}_{p^n})_{(p)}$ for $i \geq 0$ and perform an induction on i, the case of $i = 1$ being true by (i), and a basis of $\mathfrak{r}\Lambda$ being given by $(t\xi_{p^n,0}, \xi_{p^n,1}, \ldots, \xi_{p^n,p^n-1})$. Note that for $0 \leq b \leq a$, the binomial coefficient $\binom{a}{b}$ has valuation $p^{n-1}(q_p(a-b) + q_p(b) - q_p(a))$ at t.

We claim that $\mathfrak{r}\cdot\mathfrak{r}^{i-1}$ is contained in the span of the given tuple.

The set $t\xi_{p^n,0}\cdot\mathfrak{r}^{i-1}$ is contained in this span. Moreover, by (3.21), we obtain in case $1 \le j'$, $0 \le j \le j'$,

$$(*) \qquad \xi_{p^n,j'}(t^{\underline{i-1-q_p(j)}}\xi_{p^n,j}) = t^{\underline{i-1-q_p(j)}} \sum_{s\in[0,j]} \binom{j}{s}\binom{j+j'-s}{j}(-1)^s t^s \xi_{p^n,j+j'-s}.$$

So it suffices to show the inequality

$$\begin{aligned} v_t(\tbinom{j+j'-s}{j}) + \underline{i-1-q_p(j)} + s \ &\ge\ \underline{i-q_p(j+j'-s)} \\ &=\ \underline{i-q_p(j)-q_p(j'-s)+p^{1-n}v_t(\tbinom{j+j'-s}{j})} \end{aligned}$$

for $s\in[0,j]$, whence in turn it suffices to show that

$$(1-p^{1-n})v_t(\tbinom{j+j'-s}{j}) - 1 + s + q_p(j'-s) \ge 0.$$

But we have $s \ge 1$ or ($s = 0$ and $q_p(j') \ge 1$).

Furthermore, (3.21) yields

$$(**) \qquad \xi_{p^n,j'}(t^{\underline{i-1-q_p(j)}}\xi_{p^n,j}) = t^{\underline{i-1-q_p(j)}} \sum_{s\in[0,j']} \binom{j'}{s}\binom{j+j'-s}{j'}(-1)^s t^s \xi_{p^n,j+j'-s}.$$

in case $1 \le j' \le j$. So it suffices to show that

$$v_t(\tbinom{j'}{s}\tbinom{j+j'-s}{j'}) + \underline{i-1-q_p(j)} + s \ge \underline{i-q_p(j+j'-s)}$$

for $s\in[0,j']$, whence in turn, dropping a factor p^{n-1}, it suffices to see that

$$\Big((q_p(j'-s)+q_p(s)-q_p(j'))+(q_p(j-s)+q_p(j')-q_p(j+j'-s))\Big)-q_p(j)+q_p(j+j'-s)+s-1 \ge 0.$$

For $s \ge 1$, we see that both in case $(*)$ and in case $(**)$, the corresponding summand is even contained in $t\mathfrak{r}^{i-1}$.

We claim that $\mathfrak{r}\cdot\mathfrak{r}^{i-1}$ contains the given tuple.

First, we note that $t^{\underline{i-q_p(0)}}\xi_{p^n,0} = (t\xi_{p^n,0})^i$. Moreover, by the summandwise argument above concerning $(**)$, we dispose of the congruence

$$\xi_{p^n,j'}(t^{\underline{i-1-q_p(j)}}\xi_{p^n,j}) \equiv_{t\mathfrak{r}^{i-1}} t^{\underline{i-1-q_p(j)}}\binom{j+j'}{j'}\xi_{p^n,j+j'}.$$

in case $1 \le j' \le j$. In particular, given $j \in [1,p^n-1]$ such that $j[p] \le j - j[p]$, i.e. such that j is not a power of p, we obtain

$$\xi_{p^n,j[p]}(t^{\underline{i-1-q_p(j-j[p])}}\xi_{p^n,j-j[p]}) \equiv_{t\mathfrak{r}^{i-1}} t^{\underline{i-q_p(j)}}\binom{j}{j[p]}\xi_{p^n,j}\ ,$$

and it remains to note that $v_p(\binom{j}{j[p]}) = 0$.

Furthermore, for $m\in[0,n-1]$ we have

$$\xi_{p^n,p^m}(t^{\underline{i-1-q_p(0)}}\xi_{p^n,0}) = t^{\underline{i-q_p(p^m)}}\xi_{p^n,p^m}.$$

Ad (iii). Concerning the dimension of the ith radical layer, we calculate

$$\begin{aligned} \#\{k\in[0,p^n-1] \mid q_p(k)=j\} \ &=\ \#\{(a_l)_{l\in[0,n-1]}\in[0,p-1]^n \mid \textstyle\sum_{l\in[0,n-1]} a_l = j\} \\ &=\ \binom{n}{j}_{p-1}. \end{aligned}$$

Remark 4.3 The inequality mentioned in the introduction to this section reads $l_{p^n,1} \cdot l_{p^n,i} \geq l_{p^n,i+1}$ for $i \geq 1$. By (4.2 iii), this translates into the assertion that

$$(n+1)\left(\sum_{j\in[0,i]} \tbinom{n}{j}_{p-1}\right) \geq \sum_{j\in[0,i+1]} \tbinom{n}{j}_{p-1}\,.$$

For p large, this follows from

$$\begin{array}{rcl}(n+1)\left(\sum_{j\in[0,i]}\binom{n}{j}_{p-1}\right) & = & (n+1)\left(\sum_{j\in[0,i]}\binom{n+j-1}{n-1}\right) \\ & = & (n+1)\binom{n+i}{n} \\ & \geq & \frac{n+i+1}{i+1}\cdot\frac{(n+i)!}{n!i!} \\ & = & \binom{n+i+1}{n} \\ & = & \sum_{j\in[0,i+1]}\binom{n}{j}_{p-1}\,. \end{array}$$

Example 4.4 *The sequence $(l_{81,i})_{i\geq 0}$ is given by*

$$1,\ 5,\ 15,\ 31,\ 50,\ 66,\ 76,\ 80,\ 81,\ 81,\ \ldots$$

5 The absolute cyclic Wedderburn embedding

Once given the Kervaire-Murthy pullback, a closed formula describing the image of the absolute Wedderburn embedding can be derived in a straightforward manner (5.14, 5.18).

5.1 An inversion formula for $\mathbf{Q}C_{p^n}$

Let p be a prime and let $n \geq 1$.

Lemma 5.1 *The inverse of the absolute Wedderburn isomorphism*

$$\begin{array}{rcl}\mathbf{Q}C_{p^n} & \xrightarrow[\sim]{\omega_{\mathbf{Q},p^n}} & \prod_{k\in[0,n]}\mathbf{Q}[\zeta_{p^k}] \\ c_{p^n} & \longmapsto & (\zeta_{p^k})_{k\in[0,n]}\end{array}$$

is given by

$$\begin{array}{rcl}\prod_{k\in[0,n]}\mathbf{Q}[\zeta_{p^k}] & \xrightarrow[\sim]{\omega^{-1}_{\mathbf{Q},p^n}} & \mathbf{Q}C_{p^n} \\ (\sum_{i\in\mathbf{Z}} y_{k,i}\zeta^i_{p^k})_{k\in[0,n]} & \longmapsto & p^{-n}\sum_{i\in\mathbf{Z}}\left(\sum_{k\in[0,n]} p^k y_{k,i}\sum_{j\in[1,p^{n-k}]} c_{p^n}^{i+jp^k} - \sum_{k\in[1,n]} p^{k-1} y_{k,i}\sum_{j\in[1,p^{n-k+1}]} c_{p^n}^{i+jp^{k-1}}\right) \\ & = & p^{-n}\sum_{i\in\mathbf{Z}} c^i_{p^n}\sum_{k\in[0,n]} p^k \sum_{j\in[1,p^{n-k}]}\left(y_{k,i-jp^k} - y_{k+1,i-jp^k}\right),\end{array}$$

where $y_{k,i}\in\mathbf{Q}$ and where $y_{n+1,i} := 0$ for $i\in\mathbf{Z}$. We allow ‘non-reduced’ expressions for elements in $\mathbf{Q}(\zeta_{p^k})$, merely requiring $y_{k,i} = 0$ for all but finitely many $i \in \mathbf{Z}$.

We denote the absolute Wedderburn embedding, i.e. the restriction of $\omega_{\mathbf{Q},p^n}$ to $\mathbf{Z}C_{p^n}$, by

$$\mathbf{Z}C_{p^n} \xrightarrow{\omega_{\mathbf{Z},p^n}} \prod_{k\in[0,n]} \mathbf{Z}[\zeta_{p^k}].$$

By (3.6), the inversion formula over $\mathbf{Q}(\zeta_{p^n})$ reads

$$\begin{array}{rcl} \mathbf{Q}(\zeta_{p^n})C_{p^n} & \xrightarrow{\sim} & \prod\limits_{i\in[1,p^n]} \mathbf{Q}(\zeta_{p^n}) \\ c_{p^n} & \longmapsto & (\zeta_{p^n}^i)_{i\in[1,p^n]} \\ p^{-n} \sum\limits_{j\in[1,p^n]} c_{p^n}^j \sum\limits_{i\in[1,p^n]} y_i \zeta_{p^n}^{-ij} & \longleftarrow\!\shortmid & (y_i)_i. \end{array}$$

We precompose the inverse direction with the direct product of the maps

$$\begin{array}{rcl} \mathbf{Q}(\zeta_{p^k}) & \longrightarrow & \prod\limits_{\sigma\in\mathrm{Gal}(\mathbf{Q}(\zeta_{p^k})/\mathbf{Q})} \mathbf{Q}(\zeta_{p^n}) \\ x & \longmapsto & (x\sigma)_\sigma \end{array}$$

over $k \in [0,n]$, where the position of the Galois automorphism corresponding to $i \in (\mathbf{Z}/p^k)^*$, represented by $i \in [1,p^k]$, is to be identified with the position $ip^{n-k} \in [1,p^n]$.

If $l \in [1,n]$, the image of $(\partial_{l,k})_{k\in[0,n]} \in \prod_{k\in[0,n]} \mathbf{Z}[\zeta_{p^k}]$ under this composition is

$$\begin{aligned} & p^{-n} \sum_{j\in[1,p^n]} c_{p^n}^j \sum_{i\in[1,p^n],\, v_p(i)\,=\,n-l} \zeta_{p^n}^{-ij} \\ = \;& p^{-n} \sum_{j\in[1,p^n]} c_{p^n}^j \sum_{i\in(\mathbf{Z}/p^l)^*} \zeta_{p^n}^{-ijp^{n-l}} \\ = \;& p^{-n} \left(\sum_{j\in[1,p^n],\, v_p(j)\ge l} c_{p^n}^j (p-1)p^{l-1} + \sum_{j\in[1,p^n],\, v_p(j)\,=\,l-1} c_{p^n}^j (-1)p^{l-1} \right) \\ = \;& p^{-n} \left(p^l \left(\sum_{j\in[1,p^{n-l}]} c_{p^n}^{jp^l} \right) - p^{l-1} \left(\sum_{j\in[1,p^{n-l+1}]} c_{p^n}^{jp^{l-1}} \right) \right). \end{aligned}$$

If $l = 0$, it is $p^{-n} \sum\limits_{j\in[1,p^n]} c_{p^n}^j$. The inversion formula follows by $\mathbf{Q}C_{p^n}$-linearity of $\omega_{\mathbf{Q},p^n}$.

5.2 An inversion formula for $\mathbf{Q}C_m$

Let $m \ge 1$. Writing $d|m$ stands for $d \ge 1$ and $m \in (d)$. The letters p and q are reserved to denote prime numbers. We denote by $k[p'] := k/k[p]$ the p'-part of an integer $k \ge 1$. Given a finite family of groups $(G_i)_{i\in[1,l]}$, $l \ge 1$, there is a ring isomorphism

$$\begin{array}{rcl} \mathbf{Z}[\prod\limits_{i\in[1,l]} G_i] & \xrightarrow{\sim} & \bigotimes\limits_{i\in[1,l]} \mathbf{Z}G_i \\ (g_i)_i & \longmapsto & (g_i)_i^{\otimes} := g_1 \otimes \cdots \otimes g_l. \end{array}$$

For $p|k$ we denote by $s_{k,p}$ a representative in $\mathbf{Z}$ of the inverse of $k[p']$ in $(\mathbf{Z}/k[p])^*$ to obtain

$$\begin{array}{rcl} C_k & \xrightarrow{\sim} & \prod_{p|k} C_{k[p]} \\ c_k & \longmapsto & (c_{k[p]})_{p|k} \\ c_k^{\sum_{p|k} j_p s_{k,p} k[p']} & \longleftarrow & (c_{k[p]}^{j_p})_{p|k}, \end{array}$$

and

$$\begin{array}{rcl} \mathbf{Z}[\zeta_k] & \xleftarrow{\sim} & \bigotimes_{p|k} \mathbf{Z}[\zeta_{k[p]}] \\ \zeta_k^{\sum_{p|k} s_{k,p} j_p k[p']} & \longleftarrow & (\zeta_{k[p]}^{j_p})^{\otimes}_{p|k} \\ \zeta_k & \longmapsto & (\zeta_{k[p]})^{\otimes}_{p|k}. \end{array}$$

We use these isomorphisms as identifications.

We consider the absolute Wedderburn isomorphism

$$\begin{array}{rcl} \mathbf{Q}C_m & \xrightarrow[\sim]{\omega_{\mathbf{Q},m}} & \prod_{d|m} \mathbf{Q}(\zeta_d) \\ c_m & \longmapsto & (\zeta_d)_{d|m} \end{array}$$

and its restriction, the absolute Wedderburn embedding

$$\mathbf{Z}C_m \xrightarrow{\omega_{\mathbf{Z},m}} \prod_{d|m} \mathbf{Z}[\zeta_d].$$

Remark 5.2 *The embedding $\omega_{\mathbf{Z},m}$ identifies with $\bigotimes_{p|m} \omega_{\mathbf{Z},m[p]}$ in the sense that the identifications yield an isomorphism of embeddings*

$$\begin{tikzcd} \mathbf{Z}C_m \arrow[r, "\omega_{\mathbf{Z},m}"] \arrow[d] & \prod_{d|m} \mathbf{Z}[\zeta_d] \\ \bigotimes_{p|m} \mathbf{Z}C_{m[p]} \arrow[r, "\bigotimes_{p|m} \omega_{\mathbf{Z},m[p]}"] & \bigotimes_{p|m} \prod_{e|m[p]} \mathbf{Z}[\zeta_e]. \arrow[u] \end{tikzcd}$$

On both ways, c_m is mappped to $(\zeta_d)_{d|m}$.

We factorize $\omega_{\mathbf{Q},m}^{-1}$ along identifications u and w into

$$\prod_{d|m} \mathbf{Q}(\zeta_d) \xrightarrow[\sim]{u} \bigotimes_{p|m} \left(\prod_{d|m[p]} \mathbf{Q}[\zeta_d] \right) \xrightarrow[\sim]{v} \bigotimes_{p|m} \mathbf{Q}C_{m[p]} \xrightarrow[\sim]{w} \mathbf{Q}C_m,$$

v being the tensor product of the inverses of the absolute Wedderburn isomorphism belonging to the respective prime part (cf. 5.2).

For $e|m$ and a prime $p|m$, we abbreviate the element

$$\mathbf{Q}C_m \ni f_{e,p}(c_m) := \begin{cases} \frac{1}{m[p]} \sum\limits_{i\in\mathbf{Z}/m[p]} c_m^i & \text{for } v_p(e)=0 \\ \frac{e[p]}{m[p]}\left(\sum\limits_{i\in\mathbf{Z}/(\frac{m[p]}{e[p]})} c_m^{ie[p]} - \frac{1}{p} \sum\limits_{i\in\mathbf{Z}/(p\frac{m[p]}{e[p]})} c_m^{ie[p]/p} \right) & \text{for } v_p(e)\geq 1. \end{cases}$$

Suppose given an element $a_d(c_m) = \sum\limits_{i\in\mathbf{Z}/m} a_{d,i}c_m^i \in \mathbf{Q}C_m$ for each $d|m$, representing $a_d(\zeta_d) = \sum_{i\in\mathbf{Z}/m} a_{d,i}\zeta_d^i$. Using $\mathbf{Q}C_m$-linearity, we obtain

$$\begin{array}{rcl} (a_d(\zeta_d))_{d|m}uvw & = & \sum_{e|m} a_e(c_m)\cdot(\partial_{e,d})_{d|m}uvw \\ & = & \sum_{e|m} a_e(c_m)\cdot\left((\partial_{e[p],d})_{d|m[p]}\right)^{\otimes}_{p|m} vw \\ & \stackrel{(5.1)}{=} & \sum_{e|m} a_e(c_m)\cdot\left(f_{e,p}(c_{m[p]})\right)^{\otimes}_{p|m} w \\ & = & \sum_{e|m} a_e(c_m)\prod_{p|m} f_{e,p}(c_m^{s_{m,p}m[p']}). \end{array}$$

We write

$$f_{e,p}(c_m) =: \sum_{j\in\mathbf{Z}/m[p]} f_{e,p,j}c_m^j$$

and continue to calculate

$$\begin{array}{rcl} \sum_{e|m} a_e(c_m)\prod_{p|m} f_{e,p}(c_m^{s_{m,p}m[p']}) & = & \sum_{e|m}\sum_{i\in\mathbf{Z}/m} a_{e,i}c_m^i \prod_{p|m}\sum_{j\in\mathbf{Z}/m[p]} f_{e,p,j}c_m^{js_{m,p}m[p']} \\ & = & \sum_{e|m}\sum_{i\in\mathbf{Z}/m} a_{e,i}c_m^i \sum_{k\in\mathbf{Z}/m} c_m^k \prod_{p|m} f_{e,p,k} \\ & = & \sum_{l\in\mathbf{Z}/m} c_m^l \left[\sum_{e|m}\sum_{k\in\mathbf{Z}/m} a_{e,l-k}\prod_{p|m} f_{e,p,k}\right] \end{array}$$

If A is a condition which might be true or not, we let the expression $\{\text{if } A\}$ take the value 1 if A holds, and 0 if A does not hold. Given $d|m$, we let $d'|m$ be defined by

$$v_p(d') := \max(v_p(d)-1, 0)$$

for $p|m$. We rewrite

$$f_{e,p,k} = \tfrac{e[p]}{m[p]}\left[\{\text{if } e[p]|k\} - \tfrac{1}{p}\{\text{if } p|e[p]\}\{\text{if } e[p]|pk\}\right]$$

to obtain

$$\begin{array}{rcl} \prod_{p|m} f_{e,p,k} & = & \frac{e}{m}\prod_{p|m}\left[\{\text{if } e[p]|k\} - \frac{1}{p}\{\text{if } p|e[p]\}\{\text{if } e[p]|pk\}\right] \\ & = & \{\text{if } e'|k\}\cdot\frac{e}{m}\left[\prod\limits_{p|e,\ e[p]=pk[p]}\left(-\frac{1}{p}\right)\right]\left[\prod\limits_{p|e,\ e[p]|k[p]}\left(1-\frac{1}{p}\right)\right]. \end{array}$$

Proposition 5.3 (inversion formula) *The inverse of the absolute Wedderburn isomorphism*

$$\begin{array}{rcl} \mathbf{Q}C_m & \xrightarrow[\sim]{\omega_{\mathbf{Q},m}} & \prod\limits_{d|m}\mathbf{Q}(\zeta_d) \\ c_m & \longmapsto & (\zeta_d)_{d|m} \end{array}$$

maps $(a_d(\zeta_d))_{d|m}$*, written using representing elements* $a_d(c_m) = \sum_{i\in\mathbf{Z}/m} a_{d,i}c_m^i \in \mathbf{Q}C_m$*, to*

$$\frac{1}{m}\sum_{l\in\mathbf{Z}/m} c_m^l \left[\sum_{d|m} d \sum_{k\in\mathbf{Z}/(\frac{m}{d'})} a_{d,l-kd'} \left[\prod_{p|d,\, p\nmid k}\left(-\frac{1}{p}\right)\right]\left[\prod_{p|d,\, p|k}\left(1-\frac{1}{p}\right)\right]\right],$$

where $v_p(d') := \max(v_p(d)-1,0)$.

But these inversion formulas (5.1, 5.3) are not suited for giving convenient ties that describe the image of the absolute Wedderburn embedding, for in neither case they yield a *triangular* system of ties, i.e. of congruences of tuple entries.

5.3 The Kervaire-Murthy pullback

Let p be a prime and let $n \geq 1$.

Lemma 5.4 *Let* $k \in [1,n]$*. Writing*

$$\begin{aligned} f_k(X) &:= -\sum_{i\in[0,p-2]}(p-1-i)X^{ip^{k-1}} \\ g_k(X) &:= \sum_{i\in[0,p^{n-k}-1]}\left(-(p^{n-k+1}-p(i+1))X^{ip^k+p^{k-1}} + (p^{n-k+1}-p(i+1)+1)X^{ip^k}\right), \end{aligned}$$

we obtain

$$f_k(X)\cdot\left(\prod_{i\in[0,n]\setminus\{k\}}\Phi_{p^i}(X)\right) + g_k(X)\cdot\Phi_{p^k}(X) = p^{n-k+1}.$$

We expand

$$\begin{aligned} & f_k(X)\cdot\left(\prod_{i\in[0,n]\setminus\{k\}}\Phi_{p^i}(X)\right) \\ = & -\left(\textstyle\sum_{i\in[0,p-2]}(p-1-i)X^{ip^{k-1}}\right)\left((X^{p^{k-1}}-1)\sum_{i\in[0,p^{n-k}-1]}X^{ip^k}\right) \\ = & -\left(\textstyle\sum_{i\in[1,p-1]}(p-i)X^{ip^{k-1}} - \sum_{i\in[0,p-2]}(p-1-i)X^{ip^{k-1}}\right)\left(\sum_{i\in[0,p^{n-k}-1]}X^{ip^k}\right) \\ = & -\left(-p+\textstyle\sum_{i\in[0,p-1]}X^{ip^{k-1}}\right)\left(\sum_{i\in[0,p^{n-k}-1]}X^{ip^k}\right) \\ = & \ p\left(\textstyle\sum_{i\in[0,p^{n-k}-1]}X^{ip^k}\right) - \left(\sum_{i\in[0,p^{n-k+1}-1]}X^{ip^{k-1}}\right). \end{aligned}$$

Then we calculate

$$
\begin{array}{rl}
& g_k(X)\cdot\Phi_{p^k}(X) \\
= & \left(\sum_{i\in[0,p^{n-k}-1]}\left(-(p^{n-k+1}-p(i+1))X^{ip^k+p^{k-1}}+(p^{n-k+1}-p(i+1)+1)X^{ip^k}\right)\right)\cdot \\
& \cdot\left(\sum_{j\in[0,p-1]}X^{jp^{k-1}}\right) \\
= & -\sum_{i\in[0,p^{n-k}-1]}\sum_{j\in[1,p]}(p^{n-k+1}-p(i+1))X^{ip^k+jp^{k-1}} \\
& +\sum_{i\in[0,p^{n-k}-1]}\sum_{j\in[0,p-1]}(p^{n-k+1}-p(i+1)+1)X^{ip^k+jp^{k-1}} \\
= & -\sum_{i\in[0,p^{n-k}-1]}(p^{n-k+1}-p(i+1))X^{(i+1)p^k}+\sum_{i\in[0,p^{n-k}-1]}\sum_{j\in[1,p-1]}X^{ip^k+jp^{k-1}} \\
& +\sum_{i\in[0,p^{n-k}-1]}(p^{n-k+1}-p(i+1)+1)X^{ip^k} \\
= & -\sum_{i\in[1,p^{n-k}-1]}(p^{n-k+1}-pi)X^{ip^k}+\sum_{i\in[0,p^{n-k+1}-1]}X^{ip^{k-1}} \\
& +\sum_{i\in[0,p^{n-k}-1]}(p^{n-k+1}-p(i+1))X^{ip^k} \\
= & -p\left(\sum_{i\in[1,p^{n-k}-1]}X^{ip^k}\right)+\left(\sum_{i\in[0,p^{n-k+1}-1]}X^{ip^{k-1}}\right)+(p^{n-k+1}-p).
\end{array}
$$

Lemma 5.5 *Writing*

$$g_0(X) := -\sum_{i\in[0,p^n-2]}(p^n-1-i)X^i,$$

we obtain

$$\left(\prod_{i\in[0,n]\setminus\{0\}}\Phi_{p^i}(X)\right)+g_0(X)\cdot\Phi_{p^0}(X)=p^n.$$

We expand

$$
\begin{array}{rcl}
g_0(X)\cdot\Phi_{p^0}(X) & = & -\left(\sum_{i\in[0,p^n-2]}(p^n-1-i)X^i\right)(X-1) \\
& = & -\left(\sum_{i\in[1,p^n-1]}(p^n-i)X^i\right)+\left(\sum_{i\in[0,p^n-1]}(p^n-1-i)X^i\right) \\
& = & -\left(\sum_{i\in[0,p^n-1]}X^i\right)+p^n.
\end{array}
$$

Proposition 5.6 (Kervaire, Murthy [KM 77, §1]) *Let $k\geq 1$. There is a pullback diagram of rings*

$$
\begin{tikzcd}
c_{p^k} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & c_{p^{k-1}} \arrow[ddd, mapsto] \\
& \mathbf{Z}C_{p^k} \arrow[r] \arrow[d, "\alpha_k"'] \arrow[dr, phantom, "\ulcorner"] & \mathbf{Z}C_{p^{k-1}} \arrow[d] & \\
& \mathbf{Z}[\zeta_{p^k}] \arrow[r] & \mathbf{F}_pC_{p^{k-1}} & \\
\zeta_{p^k} \arrow[rrr, mapsto] & & & c_{p^{k-1}}.
\end{tikzcd}
$$

Given a commutative ring A containing ideals $\mathfrak{a}$ and $\mathfrak{b}$, there is a pullback diagram

$$
\begin{tikzcd}
A/\mathfrak{a}\cap\mathfrak{b} \arrow[r] \arrow[d] & A/\mathfrak{b} \arrow[d] \\
A/\mathfrak{a} \arrow[r] & A/\mathfrak{a}+\mathfrak{b}.
\end{tikzcd}
$$

Hence, letting $A = \mathbf{Z}[X]$, $\mathfrak{a} = (\Phi_{p^k}(X))$, $\mathfrak{b} = (X^{p^{k-1}} - 1)$, we need to show that $(\Phi_{p^k}(X)) \cap (X^{p^{k-1}} - 1) = (X^{p^k} - 1)$ and that $(\Phi_{p^k}(X), X^{p^{k-1}} - 1) = (p, X^{p^{k-1}} - 1)$. The intersection follows from the analoguous assertion in $\mathbf{Q}[X]$, using $\mathbf{Z}[X]f(X) = \mathbf{Z}[X] \cap \mathbf{Q}[X]f(X)$ for a monic polynomial $f(X) \in \mathbf{Z}[X]$. The sum follows by

$$(\Phi_{p^k}(X), X^{p^{k-1}} - 1) \overset{(5.4)}{=} (p, \Phi_{p^k}(X), X^{p^{k-1}} - 1) = (p, (X-1)^{p^k - p^{k-1}}, (X-1)^{p^{k-1}}).$$

Corollary 5.7 *The index of the absolute Wedderburn embedding $\omega_{\mathbf{Z},p^n}$ is given by*

$$\# \prod_{k \in [1,n]} \mathbf{F}_p C_{p^{k-1}} = p^{\frac{p^n - 1}{p-1}}.$$

Corollary 5.8 *For $m \geq 1$, the index of the absolute Wedderburn embedding $\omega_{\mathbf{Z},m}$ is given by*

$$\prod_{p|m} p^{\frac{m[p]-1}{p-1} \cdot m[p']}.$$

This follows from (5.7) using (5.2).

Remark 5.9 For $k \geq 1$, we let Δ_k denote the absolute value of the discriminant of $\mathbf{Z}[\zeta_k]$ over $\mathbf{Z}$. We take from [Kü 99, S 1.1.2] that the index of $\omega_{\mathbf{Z},m}$ is given by

$$\sqrt{\frac{m^m}{\prod_{d|m} \Delta_d}}.$$

A comparison with (5.8) allows to re-calculate Δ_m. First we remark that the inverse of the Dedekind isomorphism

$$\begin{array}{ccccc} \mathbf{Q}(\zeta_m) & \otimes & \mathbf{Q}(\zeta_m) & \xrightarrow[\sim]{\delta_m} & \prod_{j \in (\mathbf{Z}/m)^*} \mathbf{Q}(\zeta_m) \\ \zeta_m^k & \otimes & \zeta_m^l & \longmapsto & (\zeta_m^k \zeta_m^{jl})_{j \in (\mathbf{Z}/m)^*}, \end{array}$$

where $k, l \in [0, \varphi(m) - 1]$, is given by

$$\begin{array}{rcl} \prod_{j \in (\mathbf{Z}/m)^*} \mathbf{Q}(\zeta_m) & \xrightarrow{\delta_m^{-1}} & \mathbf{Q}(\zeta_m) \otimes \mathbf{Q}(\zeta_m) \\ (y_j)_{j \in (\mathbf{Z}/m)^*} & \longmapsto & m^{-1} \sum_{i \in \mathbf{Z}/m} \sum_{j \in (\mathbf{Z}/m)^*} y_j \zeta_m^{-ij} \otimes \zeta_m^i. \end{array}$$

In particular, its restriction to the Dedekind embedding

$$\mathbf{Z}[\zeta_m] \otimes \mathbf{Z}[\zeta_m] \overset{\delta_m}{\hookrightarrow} \prod_{j \in (\mathbf{Z}/m)^*} \mathbf{Z}[\zeta_m]$$

has a cokernel annihilated by m. In particular, the prime divisors of Δ_m divide m. We infer that for $p|m$

$$\sum_{d|m,\, v_p(d) \geq 1} v_p(\Delta_d) = \sum_{d|m} v_p(\Delta_d) \overset{(5.8)}{=} m[p'] \left(v_p(m) m[p] - 2 \cdot \frac{m[p]-1}{p-1} \right).$$

Given $d \geq 1$, $p|d$, we denote

$$u(p,d) := \frac{d[p]}{p} \Big(v_p(d)(p-1) - 1 \Big) \cdot \varphi(d[p'])$$

and obtain

$$\begin{aligned}\sum_{d|m,\, v_p(d)\geq 1} u(p,d) &= \sum_{d|m,\, v_p(d)\geq 1} \tfrac{d[p]}{p}\Big(v_p(d)(p-1)-1\Big)\cdot\varphi(d[p'])\\ &= \sum_{i\in[1,v_p(m)]} p^{i-1}\Big(i(p-1)-1\Big)\cdot\sum_{d|m[p']}\varphi(d)\\ &= \Big(v_p(m)m[p]-2\tfrac{m[p]-1}{p-1}\Big)\cdot m[p']\,,\end{aligned}$$

whence by induction on m, we get $v_p(\Delta_m) = u(p,m)$, i.e. the absolute value of the discriminant is

$$\Delta_m = \prod_{p|m} p^{\frac{m[p]}{p}\Big(v_p(m)(p-1)-1\Big)\varphi(m[p'])}\,.$$

5.4 Kervaire-Murthy ties for $\mathbf{Z}C_{p^n}$

Let p be a prime and let $n \geq 0$. Given $j\in\mathbf{Z}$ and $k\geq 1$, we let $[j]_k\in[0,k-1]$ be defined by $[j]_k \equiv_k j$. For $a,b,c\in\mathbf{Z}$ and $a\leq b$, we let $\chi_{[a,b]}(c) := 1$ if $c\in[a,b]$ and $\chi_{[a,b]}(c):=0$ if $c\notin[a,b]$.

Let $\mathfrak{l}^{\mathrm{f}}(\mathbf{Z})$ be the abelian group consisting of sequences $x=(x_j)_{j\in\mathbf{Z}}$ with entries $x_j\in\mathbf{Z}$ such that the support

$$\underline{x} := \{i\in\mathbf{Z}\mid x_i\neq 0\}\subseteq\mathbf{Z}$$

of positions carrying nonzero entries is finite. For $i\geq 0$, $m\geq 0$ and $s\geq 0$ we shall define a $\mathbf{Z}$-linear operator $T_i^{m,s}:\mathfrak{l}^{\mathrm{f}}(\mathbf{Z})\longrightarrow\mathfrak{l}^{\mathrm{f}}(\mathbf{Z})$ (which we write on the *left*).

Suppose given $j\in\mathbf{Z}$. If $i\geq m$, we let

$$\Big(T_i^{m,s}x\Big)_j := \chi_{[0,p^i-1]}(j)\cdot\sum_{k\in\mathbf{Z}} x_{[j]_{p^{i-m}}-p^{i-m}+kp^{i+s}}\,.$$

If $i<m$, we let

$$\Big(T_i^{m,s}x\Big)_j := 0.$$

We note that

$$\Big(T_i^{0,0}x\Big)_j := \chi_{[0,p^i-1]}(j)\cdot\sum_{k\in\mathbf{Z}} x_{j+kp^i}\,.$$

Lemma 5.10

(i) Given $a,b,c,d,s,t\geq 0$ such that $b-a\leq d-c\leq b+s\leq d$. Then

$$T_b^{a,s}\circ T_d^{c,t} = p^{d-b-s}\,T_b^{a,d-b+t}.$$

(ii) Given $a,b,d,s\geq 0$ such that $b+s\leq d$. Then

$$T_b^{a,s}\circ T_d^{0,0} = T_b^{a,s}.$$

Ad (i). We may assume $0 \le b-a$. Given $j \in \mathbf{Z}$, we obtain

$$\begin{aligned}
\left(T_b^{a,s}T_d^{c,t}x\right)_j &= \left(T_b^{a,s}\left(\chi_{[0,p^d-1]}(j')\sum_{k\in\mathbf{Z}} x_{[j']_{p^{d-c}}-p^{d-c}+kp^{d+t}}\right)_{j'\in\mathbf{Z}}\right)_j \\
&= \chi_{[0,p^b-1]}(j)\cdot\sum_{k'\in\mathbf{Z}}\chi_{[0,p^d-1]}([j]_{p^{b-a}}-p^{b-a}+k'p^{b+s})\cdot \\
&\quad \cdot\sum_{k\in\mathbf{Z}} x_{[[j]_{p^{b-a}}-p^{b-a}+k'p^{b+s}]_{p^{d-c}}-p^{d-c}+kp^{d+t}} \\
&= \chi_{[0,p^b-1]}(j)p^{d-b-s}\sum_{k\in\mathbf{Z}} x_{[j]_{p^{b-a}}-p^{b-a}+kp^{d+t}} \\
&= \left(p^{d-b-s}T_b^{a,d-b+t}x\right)_j .
\end{aligned}$$

Ad (ii). The operator $T_b^{a,s}$ is invariant under sequence shifts by p^{b+s}, hence under sequence shifts by p^d.

Lemma 5.11 *Given $m \ge 1$, $l \in [1,m]$ and $a \ge 0$, we obtain*

$$\sum_{i\in[0,l-1]} p^{l-1-i}T_{m-l}^{a,l-1-i}\circ(T_{m-i}^{0,0}-T_{m-i}^{1,0}) = T_{m-l}^{a,0}-p^lT_{m-l}^{a,l}.$$

In fact,

$$\begin{aligned}
\sum_{i\in[0,l-1]} p^{l-1-i}T_{m-l}^{a,l-1-i}\circ(T_{m-i}^{0,0}-T_{m-i}^{1,0}) &\overset{(5.10\text{ i, ii})}{=} \sum_{i\in[0,l-1]} p^{l-1-i}T_{m-l}^{a,l-1-i} - \sum_{i\in[0,l-1]} p^{l-i}T_{m-l}^{a,l-i} \\
&= \sum_{i\in[1,l]} p^{l-i}T_{m-l}^{a,l-i} - \sum_{i\in[0,l-1]} p^{l-i}T_{m-l}^{a,l-i} \\
&= T_{m-l}^{a,0}-p^lT_{m-l}^{a,l}.
\end{aligned}$$

Lemma 5.12 *For $x=(x_j)_{j\in\mathbf{Z}} \in \mathfrak{l}^{\mathrm{f}}(\mathbf{Z})$ and $l \ge 0$, we denote*

$$x*\zeta_{p^l} := \sum_{j\in\mathbf{Z}} x_j\zeta_{p^l}^j$$

and obtain

$$x*\zeta_{p^l} = ((T_l^{0,0}-T_l^{1,0})x)*\zeta_{p^l}.$$

We claim that $(T_l^{1,0}x)*\zeta_{p^l}=0$. We may suppose $l \ge 1$ to calculate

$$\begin{aligned}
(T_l^{1,0}x)*\zeta_{p^l} &= \textstyle\sum_{j\in\mathbf{Z}}\chi_{[0,p^l-1]}(j)\sum_{k\in\mathbf{Z}} x_{[j]_{p^{l-1}}-p^{l-1}+kp^l}\zeta_{p^l}^j \\
&= \textstyle\sum_{k\in\mathbf{Z}}\sum_{h\in[0,p^{l-1}-1]}\sum_{i\in[0,p-1]} x_{[h+ip^{l-1}]_{p^{l-1}}-p^{l-1}+kp^l}\zeta_{p^l}^{h+ip^{l-1}} \\
&= \textstyle\sum_{k\in\mathbf{Z}}\sum_{h\in[0,p^{l-1}-1]} x_{h-p^{l-1}+kp^l}\zeta_{p^l}^h\sum_{i\in[0,p-1]}\zeta_{p^l}^{ip^{l-1}} \\
&= 0.
\end{aligned}$$

Lemma 5.13 *For $l \geq 1$, $s \geq 0$ and $j \in \mathbf{Z}\backslash[0, \varphi(p^l) - 1]$, we have $((T_l^{0,s} - T_l^{1,s})x)_j = 0$.*

In fact,

$$\begin{array}{rcl} (T_l^{1,s}x)_{(p-1)p^{l-1}+j} & = & \sum_{k\in\mathbf{Z}} x_{[(p-1)p^{l-1}+j]_{p^{l-1}} - p^{l-1} + kp^{l+s}} \\ & = & \sum_{k\in\mathbf{Z}} x_{j-p^{l-1}+kp^{l+s}} \\ & = & (T_l^{0,s}x)_{(p-1)p^{l-1}+j} \end{array}$$

for $j \in [0, p^{l-1} - 1]$.

Theorem 5.14 *The image of the absolute Wedderburn embedding is given by*

$$\begin{array}{rl} (\mathbf{Z}C_{p^n})\omega_{\mathbf{Z},p^n} = & \Bigg\{ \left(\sum\limits_{j\in[0,\varphi(p^i)-1]} x_{i,j}\zeta_{p^i}^j \right)_{i\in[0,n]},\ x_{i,j} \in \mathbf{Z} \ \Bigg| \\ & \text{for } l \in [1,n] \text{ and } j \in [0, \varphi(p^{n-l}) - 1] \text{ we have } \ x_{n-l,j} \equiv_{p^l} \sum\limits_{i\in[0,l-1]} p^{l-1-i} \cdot \\ & \cdot \sum\limits_{k\in[1,p-1]} \left(x_{n-i,j-p^{n-l}+kp^{n-1-i}} - (1 - \partial_{l,n}) x_{n-i,[j]_{p^{n-l-1}} - p^{n-l-1} + kp^{n-1-i}} \right) \Bigg\} \\ = & \Bigg\{ \left(x_i * \zeta_{p^i} \right)_{i\in[0,n]},\ x_i \in \mathfrak{l}^{\mathrm{f}}(\mathbf{Z}),\ \underline{x_i} \subseteq [0, \varphi(p^i) - 1] \ \Bigg| \\ & \text{for } l \in [1,n] \text{ we have } \ x_{n-l} \equiv_{p^l} \sum\limits_{i\in[0,l-1]} p^{l-1-i} \left(T_{n-l}^{0,l-1-i} - T_{n-l}^{1,l-1-i} \right) x_{n-i} \Bigg\} \\ \subseteq & \prod\limits_{i\in[0,n]} \mathbf{Z}[\zeta_{p^i}]. \end{array}$$

This system of ties is of triangular shape. In particular, the elementary divisors of $\omega_{\mathbf{Z},p^n}$ over $\mathbf{Z}$ are given by p^i with multiplicity $\varphi(p^{n-i})$ for $i \in [0, n]$.

The second description being true if $n = 0$, we perform an induction on n. We see by (5.6) that $\left(x_i * \zeta_{p^i}\right)_{i\in[0,n]} \in \prod_{i\in[0,n]} \mathbf{Z}[\zeta_{p^i}]$ is contained in $(\mathbf{Z}C_{p^n})\omega_{\mathbf{Z},p^n}$ if and only if the conditions (a) and (b) below hold. To formulate (b), we shall make use of (5.12).

(a) The tuple $\left(x_i * \zeta_{p^i}\right)_{i\in[0,n-1]}$ is contained in $(\mathbf{Z}C_{p^{n-1}})\omega_{\mathbf{Z},p^{n-1}}$.

(b) We have

$$\left(\left((T_i^{0,0} - T_i^{1,0}) x_n \right) * \zeta_{p^i} \right)_{i\in[0,n-1]} - \left(x_i * \zeta_{p^i} \right)_{i\in[0,n-1]} \in p(\mathbf{Z}C_{p^{n-1}})\omega_{\mathbf{Z},p^{n-1}}.$$

Condition (b) in turn holds if and only if the conditions (ba) and (bb) below hold. To formulate (ba), we use (5.13). To formulate (bb), we shall use that by induction, the description is valid in case $n - 1$.

(ba) We have $(T_i^{0,0} - T_i^{1,0})x_n \equiv_p x_i$ for $i \in [0, n-1]$.

(bb) We have

$$\begin{aligned} & -\left(T^{0,0}_{(n-1)-l} - T^{1,0}_{(n-1)-l}\right) x_n \\ & + \sum_{i\in[0,l-1]} p^{l-1-i} \left(T^{0,l-1-i}_{(n-1)-l} - T^{1,l-1-i}_{(n-1)-l}\right) \left(T^{0,0}_{(n-1)-i} - T^{1,0}_{(n-1)-i}\right) x_n \\ \equiv_{p^{l+1}} & -x_{(n-1)-l} + \sum_{i\in[0,l-1]} p^{l-1-i} \left(T^{0,l-1-i}_{(n-1)-l} - T^{1,l-1-i}_{(n-1)-l}\right) x_{(n-1)-i} \end{aligned}$$

for $l \in [1, n-1]$.

By (5.11), we may equivalently reformulate to

(bb) We have

$$\begin{aligned} 0 \equiv_{p^{l+1}} & -x_{(n-1)-l} + p^l \left(T^{0,l}_{(n-1)-l} - T^{1,l}_{(n-1)-l}\right) x_n \\ & + \sum_{i\in[0,l-1]} p^{l-1-i} \left(T^{0,l-1-i}_{(n-1)-l} - T^{1,l-1-i}_{(n-1)-l}\right) x_{(n-1)-i} \end{aligned}$$

for $l \in [1, n-1]$.

Shifting the indices i and l by one, we may in turn equivalently reformulate this assertion to

(bb) We have

$$x_{n-l} \equiv_{p^l} \sum_{i\in[0,l-1]} p^{l-1-i} \left(T^{0,l-1-i}_{n-l} - T^{1,l-1-i}_{n-l}\right) x_{n-i}$$

for $l \in [2, n]$.

Adjoining condition (ba) in case $i = n-1$, we obtain the necessity of the claimed set of ties.

Conversely, we may use (5.11) to see that

$$(T^{0,0}_{i-1} - T^{1,0}_{i-1}) \circ (T^{0,0}_{i} - T^{1,0}_{i}) \equiv_p (T^{0,0}_{i-1} - T^{1,0}_{i-1})$$

for $i \geq 1$, whence the reduction of (bb) to the modulus p suffices to conclude that (ba) holds in all its cases. Moreover, condition (a) follows by induction assumption and by reading (bb) modulo p^{l-1} for $l \in [2, n]$, thus discarding the summand for $i = 0$. Thus our claimed set of ties is also sufficient.

To see that the second description agrees with the first one, it remains to show that

$$\left(\left(T^{0,l-1-i}_{n-l} - T^{1,l-1-i}_{n-l}\right) x_{n-i}\right)_j = 0$$

for $l \in [1, n-1]$, $i \in [0, l-1]$ and $j \in \mathbf{Z}\backslash[0, \varphi(p^{n-l}) - 1]$. But this follows from (5.13).

Remark 5.15 It might be worthwhile to try to give an ad hoc proof of (5.14) by verification of the ties on the image of the canonical group basis of $\mathbf{Z}C_{p^n}$ and by a comparison of indices – which we have not attempted to do. We preferred to proceed in the straightforward manner as above since, in this way, the role of the Kervaire-Murthy pullback (5.6) remains visible.

Example 5.16 Let $p = 3$. We obtain

$$\begin{array}{lcl}
\mathbf{Z}C_3 & \xrightarrow[\sim]{\omega_{\mathbf{Z},3}} & \left\{ \left(\sum_{k\in[0,1]} x_{1,k}\zeta_3^k\right) \times x_{0,0} \;\middle|\; x_{0,0} \equiv_3 x_{1,0} + x_{1,1} \right\} \subseteq \mathbf{Z}[\zeta_3] \times \mathbf{Z} \\
\mathbf{Z}C_9 & \xrightarrow[\sim]{\omega_{\mathbf{Z},9}} & \Big\{ \left(\sum_{k\in[0,5]} x_{2,k}\zeta_9^k\right) \times \left(\sum_{k\in[0,1]} x_{1,k}\zeta_3^k\right) \times x_{0,0} \;\Big| \\
& & x_{1,0} \equiv_3 x_{2,0} + x_{2,3} - x_{2,2} - x_{2,5} \\
& & x_{1,1} \equiv_3 x_{2,1} + x_{2,4} - x_{2,2} - x_{2,5} \\
& & x_{0,0} \equiv_9 3(x_{2,2} + x_{2,5}) + (x_{1,0} + x_{1,1}) \Big\} \subseteq \mathbf{Z}[\zeta_9] \times \mathbf{Z}[\zeta_3] \times \mathbf{Z} \\
\mathbf{Z}C_{27} & \xrightarrow[\sim]{\omega_{\mathbf{Z},27}} & \Big\{ \left(\sum_{k\in[0,17]} x_{3,k}\zeta_{27}^k\right) \times \left(\sum_{k\in[0,5]} x_{2,k}\zeta_9^k\right) \times \left(\sum_{k\in[0,1]} x_{1,k}\zeta_3^k\right) \times x_{0,0} \;\Big| \\
& & x_{2,0} \equiv_3 x_{3,0} + x_{3,9} - x_{3,6} - x_{3,15} \\
& & x_{2,1} \equiv_3 x_{3,1} + x_{3,10} - x_{3,7} - x_{3,16} \\
& & x_{2,2} \equiv_3 x_{3,2} + x_{3,11} - x_{3,8} - x_{3,17} \\
& & x_{2,3} \equiv_3 x_{3,3} + x_{3,12} - x_{3,6} - x_{3,15} \\
& & x_{2,4} \equiv_3 x_{3,4} + x_{3,13} - x_{3,7} - x_{3,16} \\
& & x_{2,5} \equiv_3 x_{3,5} + x_{3,14} - x_{3,8} - x_{3,17} \\
& & x_{1,0} \equiv_9 3(x_{3,6} + x_{3,15} - x_{3,8} - x_{3,17}) + (x_{2,0} + x_{2,3} - x_{2,2} - x_{2,5}) \\
& & x_{1,1} \equiv_9 3(x_{3,7} + x_{3,16} - x_{3,8} - x_{3,17}) + (x_{2,1} + x_{2,4} - x_{2,2} - x_{2,5}) \\
& & x_{0,0} \equiv_{27} 9(x_{3,8} + x_{3,17}) + 3(x_{2,2} + x_{2,5}) + (x_{1,0} + x_{1,1}) \Big\} \\
& & \subseteq \mathbf{Z}[\zeta_{27}] \times \mathbf{Z}[\zeta_9] \times \mathbf{Z}[\zeta_3] \times \mathbf{Z}.
\end{array}$$

5.5 Kervaire-Murthy ties for $\mathbf{Z}C_m$

Let $m \geq 1$. We maintain the notation of section 5.2. Suppose given an inclusion of commutative $\mathbf{Z}$-orders $\Lambda \subseteq \Gamma$ that has, as an inclusion of abelian groups, a cokernel Γ/Λ which is finite as a set. Let the *naive localization* $\Lambda_{[p],\Gamma}$ be the kernel of the composition $\Gamma \longrightarrow \Gamma/\Lambda \longrightarrow (\Gamma/\Lambda)_{(p)}$ (cf. [Kü 99, D.2]). Since $\Lambda_{[p],\Gamma} = \Gamma \cap \Lambda_{(p)}$, intersected as subsets of $\Gamma_{(p)}$, the naive localization $\Lambda_{[p],\Gamma}$ is a suborder of Γ of index $(\#(\Gamma/\Lambda))[p]$. We have $\Lambda_{(p)} = (\Lambda_{[p],\Gamma})_{(p)}$. Moreover,

$$\Lambda = \bigcap_p \Lambda_{[p],\Gamma}.$$

Note that if $\Lambda_{(p)} = \Gamma_{(p)}$, then $\Lambda_{[p],\Gamma} = \Gamma$.

Lemma 5.17 *Given inclusions of* $\mathbf{Z}$*-orders* $\Lambda \subseteq \Gamma$ *and* $\Lambda' \subseteq \Gamma'$*, we obtain an equality*

$$(\Lambda \otimes \Lambda')_{[p],\Gamma\otimes\Gamma'} = \Lambda_{[p],\Gamma} \otimes \Lambda'_{[p],\Gamma'}.$$

as subsets of $\Gamma \otimes \Gamma'$.

In fact, flatness yields

$$\begin{array}{rcl}
\Lambda_{[p],\Gamma} \otimes \Lambda'_{[p],\Gamma'} & = & (\Lambda_{[p],\Gamma} \otimes \Lambda'_{(p)}) \cap (\Lambda_{[p],\Gamma} \otimes \Gamma') \\
& = & (\Lambda \otimes \Lambda')_{(p)} \cap (\Lambda_{(p)} \otimes \Gamma') \cap (\Gamma \otimes \Gamma') \\
& = & (\Lambda \otimes \Lambda')_{[p],\Gamma\otimes\Gamma'}.
\end{array}$$

Letting

$$\begin{array}{rcl}
(\Lambda \subseteq \Gamma) & := & \Big((\mathbf{Z}C_{m[p]})\omega_{\mathbf{Z},m[p]} \hookrightarrow \prod_{e|m[p]} \mathbf{Z}[\zeta_e] \Big) \\
(\Lambda' \subseteq \Gamma') & := & \Big((\mathbf{Z}C_{m[p']})\omega_{\mathbf{Z},m[p']} \hookrightarrow \prod_{f|m[p']} \mathbf{Z}[\zeta_f] \Big),
\end{array}$$

we obtain

$$
\begin{aligned}
(\Lambda\otimes\Lambda')_{[p],\Gamma\otimes\Gamma'} &\overset{(5.17)}{=} \Lambda_{[p],\Gamma}\otimes\Lambda'_{[p],\Gamma'} \\
&\overset{(5.8)}{=} \Lambda\otimes\Gamma' \\
&= \prod_{f|m[p']}\Lambda\otimes\mathbf{Z}[\zeta_f] \\
&\subseteq \prod_{f|m[p']}\left(\prod_{e|m[p]}\mathbf{Z}[\zeta_e]\right)\otimes\mathbf{Z}[\zeta_f] \\
&= \prod_{f|m[p']}\ \prod_{e|m[p]}\mathbf{Z}[\zeta_{ef}].
\end{aligned}
$$

We could not do better than to argue with a $\mathbf{Z}$-linear basis of $\mathbf{Z}[\zeta_f]$. This lead to the 'non-canonical' representation of an element of $\mathbf{Z}[\zeta_d]$, $d|m$, that is used in the following

Proposition 5.18 *Suppose given an element*

$$
\left(\sum_{(j_p)_{p|d}\in\prod_{p|d}[0,\varphi(d[p])-1]} a_{d,(j_p)_{p|d}}\zeta_d^{\sum_{p|d}s_{d,p}j_p d[p']}\right)_{d|m} \in \prod_{d|m}\mathbf{Z}[\zeta_d],
$$

where $a_{d,(j_p)_{p|d}}\in\mathbf{Z}$. This element is contained in the image of $\omega_{\mathbf{Z},m}$ if and only if for each $p|m$, for each $f|m[p']$ and for each tuple $(j_q)_{q|f}$, $j_q\in[0,\varphi(d[q])-1]$, we have

$$
\left(\sum_{j_p\in[0,\varphi(e)-1]} a_{ef,\,j_p\times(j_q)_{q|f}}\zeta_e^{j_p}\right)_{e|m[p]} \in (\mathbf{Z}C_{m[p]})\omega_{\mathbf{Z},m[p]},
$$

where $j_p\times(j_q)_{q|f}$ denotes the tuple that has entry j_p at position p and j_q at position q for $q|f$. Thus we may employ (5.14).

Example 5.19 We have $s_{6,2}=1$, $s_{6,3}=2$ and $s_{12,2}=3$, $s_{12,3}=1$. The element

$$
\begin{aligned}
&(a_{1,(\bullet,\bullet)})\times(a_{2,(0,\bullet)})\times(a_{3,(\bullet,0)}+a_{3,(\bullet,1)}\zeta_3)\times(a_{4,(0,\bullet)}+a_{4,(1,\bullet)}\zeta_4)\times(a_{6,(0,0)}+a_{6,(0,1)}\zeta_6^4)\\
&\times(a_{12,(0,0)}+a_{12,(0,1)}\zeta_{12}^4+a_{12,(1,0)}\zeta_{12}^9+a_{12,(1,1)}\zeta_{12}),
\end{aligned}
$$

the symbol $\bullet$ indicating a non-existing entry, is contained in $(\mathbf{Z}C_{12})\omega_{\mathbf{Z},12}$ if and only if

$$
\begin{array}{rcrcll}
(a_{1,(\bullet,\bullet)}) & \times & (a_{2,(0,\bullet)}) & \times & (a_{4,(0,\bullet)}+a_{4,(1,\bullet)}\zeta_4) & \in (\mathbf{Z}C_4)\omega_{\mathbf{Z},4}\\
(a_{3,(\bullet,0)}) & \times & (a_{6,(0,0)}) & \times & (a_{12,(0,0)}+a_{12,(1,0)}\zeta_4) & \in (\mathbf{Z}C_4)\omega_{\mathbf{Z},4}\\
(a_{3,(\bullet,1)}) & \times & (a_{6,(0,1)}) & \times & (a_{12,(0,1)}+a_{12,(1,1)}\zeta_4) & \in (\mathbf{Z}C_4)\omega_{\mathbf{Z},4}\\
 & & (a_{1,(\bullet,\bullet)}) & \times & (a_{3,(\bullet,0)}+a_{3,(\bullet,1)}\zeta_3) & \in (\mathbf{Z}C_3)\omega_{\mathbf{Z},3}\\
 & & (a_{2,(0,\bullet)}) & \times & (a_{6,(0,0)}+a_{6,(0,1)}\zeta_3) & \in (\mathbf{Z}C_3)\omega_{\mathbf{Z},3}\\
 & & (a_{4,(0,\bullet)}) & \times & (a_{12,(0,0)}+a_{12,(0,1)}\zeta_3) & \in (\mathbf{Z}C_3)\omega_{\mathbf{Z},3}\\
 & & (a_{4,(1,\bullet)}) & \times & (a_{12,(1,0)}+a_{12,(1,1)}\zeta_3) & \in (\mathbf{Z}C_3)\omega_{\mathbf{Z},3}.
\end{array}
$$

By (5.14), an element $(b_{1,0})\times(b_{2,0})\times(b_{4,0}+b_{4,1}\zeta_4)$ is contained in $(\mathbf{Z}C_4)\omega_{\mathbf{Z},4}$ if and only if

$$
\begin{aligned}
b_{2,0} &\equiv_2 b_{4,0}-b_{4,1}\\
b_{1,0} &\equiv_4 2b_{4,1}+b_{2,0},
\end{aligned}
$$

and an element $(b_{1,0})\times(b_{3,0}+b_{3,1}\zeta_3)$ is contained in $(\mathbf{Z}C_3)\omega_{\mathbf{Z},3}$ if and only if

$$
b_{1,0}\equiv_3 b_{3,0}+b_{3,1}.
$$

Remark 5.20 Kleinert gives a system of ties that describes the image of the absolute Wedderburn embedding $\mathbf{Z}C_m \xrightarrow{\omega_{\mathbf{Z},m}} \prod_{d|m} \mathbf{Z}[\zeta_d]$ in terms of certain prime ideals of $\mathbf{Z}[\zeta_d]$, $d|m$, in case m is squarefree [Kl 81, p. 550].

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